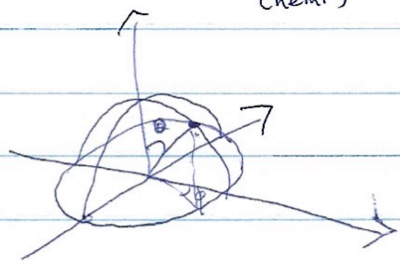
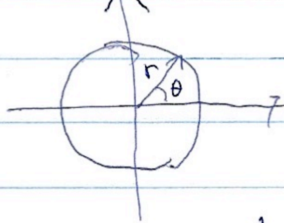
olchinski 3.1 (a)

hemispherical metric : $\underset{\text{(hemi)}}{g_{ab}} = \begin{pmatrix} r^2 & \\ & r^2 \sin^2\theta \end{pmatrix}$ in (θ, ϕ)



flat disk :



$g_{ab} = \begin{pmatrix} 1 & \\ & r^2 \end{pmatrix}$ in (r, θ)

to match the coordinates of the hemisphere, we rename (r, θ) to (θ, ϕ)

$\underset{\text{(flat)}}{g_{ab}} = \begin{pmatrix} 1 & \\ & \theta^2 \end{pmatrix}$ in (θ, ϕ)

I would like to show that the two manifolds are diff x Weyl equivalent, under $\theta \to f(\theta)$, $\phi \to g(\phi)$, $g_{ab} \to w^2 g_{ab}$,

we have

$$\underset{\text{(flat)}}{g_{ab}} = \begin{pmatrix} 1 & \\ & \theta^2 \end{pmatrix} \longrightarrow w^2 \begin{pmatrix} \left(\frac{\partial f}{\partial \theta}\right)^2 & \\ & f^2(\theta) \left(\frac{\partial g}{\partial \phi}\right)^2 \end{pmatrix}$$

write
$$\begin{cases} w^2\left(\frac{\partial f}{\partial \theta}\right)^2 = r^2 \\ w^2\left(\frac{\partial g}{\partial \phi}\right)^2 f^2(\theta) = r^2 \sin^2\theta \end{cases}$$

Ignoring sign ambiguity for now,

$$\begin{cases} w f' = r \\ w g' f = r\sin\theta \end{cases} \Rightarrow w = \frac{r}{f'}$$

$$\Rightarrow \frac{\cancel{r}}{f'} g' f = \cancel{r}\sin\theta$$

$$\frac{f}{f'} g' = \sin\theta$$

Now, $g(\phi)$ should only be a function of ϕ, thus we shall demand that g' should not contain "θ" terms that is,

$$g' = \sin\theta \frac{f'(\theta)}{f(\theta)} = \text{constant}.$$

This gives us a differential equation to solve for ~~f'(θ)~~, $f(\theta)$, it's not so difficult, letting $f(\theta) = \tan(\theta/2)$

$$\Rightarrow \sin\theta \frac{f'(\theta)}{f(\theta)} = \sin\theta \frac{\sec^2\frac{\theta}{2}}{\cancel{\sec^2\frac{\theta}{2}}} \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \sin\theta \frac{1}{\cos^2\frac{\theta}{2}} \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= 2$$

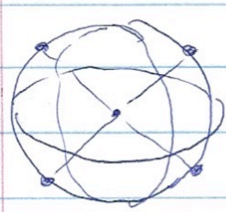
Thus we have solved f, g:

$$\boxed{\begin{cases} f(\theta) = \tan\left(\frac{\theta}{2}\right) \\ g(\phi) = 2\phi \end{cases}}$$

This coordinate transformation + Weyl brings the flat disk to the hemisphere.

$\Rightarrow$ hemisphere has the same euler # as flat disk because they are topologically equivalent

Now we determine the euler # of the flat disk. This can be done via a cross-cap of the unit sphere:



This surface has no hole, no handle, but 1 cross-cap $\Rightarrow$ $\chi = 2 - 1 = \boxed{1}$

By argument before, the hemi will have $\boxed{\chi = 1}$ as well

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