Polchinski 2-12

2.7.5 (a) The contour argument (2.6) educates that for Q_m defined by

$$Q_m = \oint_C \frac{dz}{2\pi i} z^{m+1} f(z),$$

its commutator is given by

$$[Q_m, Q_n] = \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} z_1^{m+1} f(z_1) z_2^{n+1} f(z_2)$$

$\alpha_m^\mu = \left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z)$ is identified with Q_m

by setting $f(z) = \left(\frac{2}{\alpha'}\right)^{1/2} \frac{i}{z} \partial X^\mu(z)$,

thus

$$[\alpha_m^\mu, \alpha_n^\nu] = \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} z_1^m z_2^n \left(\frac{2}{\alpha'}\right)(-1) \partial X^\mu(z_1) \partial X^\nu(z_2)$$

$$= \left(-\frac{2}{\alpha'}\right) \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} z_1^m z_2^n \partial X^\mu(z_1) \partial X^\nu(z_2)$$

$$:X^\mu(z_1) X^\nu(z_2): = X^\mu(z_1) X^\nu(z_2) + \frac{\alpha'}{2} \ln |z_{12}|^2$$

$$\Rightarrow X^\mu(z_1) X^\nu(z_2) \sim -\frac{\alpha'}{2} \ln |z_{12}|^2$$

$$\partial X^\mu(z_1)\,\partial X^\nu(z_2) \sim \left(-\frac{\alpha'}{2}\right) \partial^2 \ln|z_{12}|^2$$

$$\sim +\frac{\alpha'}{2}\frac{1}{z_{12}^2}$$

$$\Rightarrow (\cdots) = \left(-\frac{2}{\alpha'}\right)\left(-\frac{\alpha'}{2}\right) \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} z_1^m z_2^n \frac{1}{z_{12}}.$$

$$z_1^m z_2^n = (z_{12}+z_2)^m (z_1 - z_{12})^n$$

$$= \left[\sum_{k=0}^{m} \binom{m}{k} z_{12}^k z_2^{m-k}\right]\left[\sum_{l=0}^{n} \binom{n}{l} z_1^{n-l} (-z_{12})^l\right]$$

The $O(z_{12})$ terms are given by

$$k=1,\ l=0:\quad m\, z_{12}\, z_2^{m-1} z_1^n$$
$$k=0,\ l=1:\quad n(-z_{12})\, z_1^{n-1} z_2^m$$

$$\Rightarrow \operatorname{Res}_{z_1\to z_2} z_1^m z_2^n \frac{1}{z_{12}} =$$

$$= \operatorname{Res}_{z_1 \to z_2} \left(m\, z_1^n z_2^{m-1} - z_1^{n-1} z_2^m\right)\frac{1}{z_{12}}$$

$$= (m-n)\, z_2^{n+m-1}$$

Then $\oint_{C_2} \frac{dz_2}{2\pi i} (m-n) z_2^{n+m-1}$ demands $n+m=0$ for nontriviality

$\Rightarrow n = -m$

$$\boxed{= 2m\, \delta_{m,-n} \Rightarrow [\alpha_m^\mu, \alpha_n^\nu] = -2m\, \delta_{m,-n}\, \eta^{\mu\nu}}$$

Somehow I'm off by a factor of $-\frac{1}{2}$, If was in the OPE of XX?

2.7.5(b) The contour argument teaches us that for operators defined by contour integrals.

$$Q_f = \oint \frac{dz}{2\pi i} f(z),$$ its commutators are given by

$$[Q_{f_1}, Q_{f_2}] = \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} f_1(z_1) f_2(z_2).$$

The natural definition for x^μ as a contour integral is

$$x^\mu = Q_x = \oint \frac{dz}{2\pi i} X^\mu(z),$$

The definition for p^ν is given in (2.7.3)

$$p^\nu = \left(\frac{2}{\alpha'}\right)^{1/2} \alpha_0^\nu = \left(\frac{2}{\alpha'}\right) \oint \frac{dz}{2\pi} \partial X^\nu(z)$$

Thus we identify x^μ, p^ν ~~as~~ with Q_f by:

$$f_{x^\mu} = X^\mu(z) \qquad f_{p^\nu} = \left(\frac{2}{\alpha'}\right) i\,\partial X^\nu(z)$$

$$\Rightarrow [Q_{f_{x^\mu}}, Q_{f_{p^\nu}}] = [x^\mu, p^\nu]$$

$$= \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} X^\mu(z_1) \left(\frac{2}{\alpha'}\right) i\,\partial X^\nu(z_2)$$

$$= \oint_{C_2} \frac{dz_2}{2\pi i} \left(\frac{2i}{\alpha'}\right) \operatorname{Res}_{z_1 \to z_2} \left(-\frac{\alpha'}{2}\right) \partial \ln|z_{12}|^2\, \eta^{\mu\nu}$$

$$= \cancel{i\eta^{\mu\nu}} \boxed{-i\eta^{\mu\nu}}$$

⇐ Again, somehow off by -1 factor, maybe I got it opposite in OPE?

Polchinski 2.12

2.7.17 $b(z) = \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+\lambda}}$, $c(z) = \sum_{n=-\infty}^{\infty} \frac{c_m}{z^{n+1-\lambda}}$

$$\Rightarrow \quad b_m = \oint \frac{dz}{2\pi i} z^{m+\lambda-1} b(z) \qquad c_n = \oint \frac{dz}{2\pi i} z^{n-\lambda} c(z)$$

$$\{b_m, c_n\} = \oint_C \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} \left[z_1^{m+\lambda-1} b(z_1) \right] \left[z_2^{n-\lambda} c(z_2) \right].$$

$$b(z_1) c(z_2) \sim \frac{1}{z_{12}}$$

$$= \oint_C \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} \left[\frac{1}{z_{12}} z_1^{m+\lambda-1} z_2^{n-\lambda} \right]$$

$$= \oint_C \frac{dz_2}{2\pi i} z_2^{m+n-1}$$

$$= \boxed{\delta_{m,-n}}$$

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