Polchinski 1.3

$$\chi = \frac{1}{4\pi} \int_M d\tau\, d\sigma\, (-\gamma)^{1/2} R$$

Weyl: $(-\gamma')^{1/2} R' = (-\gamma)^{1/2} (R - 2\nabla^2 \omega)$.

$$\Rightarrow \Delta\chi = \frac{1}{4\pi} \int d\tau\, d\sigma\, (-\gamma)^{1/2} (-2\nabla^2 \omega)$$

$$= \frac{-1}{2\pi} \int_M d\tau\, d\sigma\, (-\gamma)^{1/2} \nabla^2 \omega .$$

write $(-\gamma)^{1/2} \nabla_a [\nabla^a \omega] = \partial_a [(-\gamma)^{1/2} (\partial^a \omega)]$

$$\Delta\chi = \frac{-1}{2\pi} \int_M d^2\sigma^a\, \partial_a [(-\gamma)^{1/2} (\partial^a \omega)]$$

$\Uparrow$ of form $\int dx^2\, \vec{\nabla} \cdot \vec{F}(\vec{x}) = \oint_S d\vec{l} \cdot \vec{F}$

$$= \frac{-1}{2\pi} \oint_{\partial M} dl^a (-\gamma)^{1/2} (\partial^a \omega)$$

$$\Rightarrow \Delta\chi = \frac{-1}{2\pi} \int_{\partial M} ds\, (\partial^a \omega)\, n_a$$

s affine parameter, proper time, n_a unit normal.

Now we consider $\Delta_{\text{weyl}} \frac{1}{2\pi} \int_{\partial M} ds\, k \quad , \quad k = t^a n_b \nabla_a t^b$

$\begin{cases} t^a \text{ unit tangent} \\ n_b \text{ unit normal} \end{cases}$

$$\nabla_a t^b = \partial_a t^b + \Gamma^b_{ac} t^c$$

$$\Gamma^b_{ac} = \frac{1}{2} g^{b\lambda} [g_{a\lambda,c} + g_{c\lambda,a} - g_{ac,\lambda}]$$

under $g_{ab} \to e^{2w} g_{ab}$, $\Gamma \to \Gamma'$

$$\Gamma^{b\,\prime}_{ac} = \frac{1}{2} e^{-2w} g^{b\lambda} [\partial_c [e^{2w} g_{a\lambda}] + \partial_a [e^{2w} g_{c\lambda}] - \partial_\lambda [e^{2w} g_{ac}]]$$

$$= \frac{1}{2} \cancel{e^{-2w}} g^{b\lambda} \{ 2(\partial_c w) \cancel{e^{2w}} g_{a\lambda} + \cancel{e^{2w}} g_{a\lambda,c} + 2(\partial_a w) \cancel{e^{2w}} g_{c\lambda} + \cancel{e^{2w}} g_{c\lambda,a} - 2(\partial_\lambda w) \cancel{e^{2w}} g_{ac} - \cancel{e^{2w}} g_{ac,\lambda} \}$$

$$\Gamma^{b\,\prime}_{ac} = \Gamma^b_{ac} + g^{b\lambda} [(\partial_c w) g_{a\lambda} + (\partial_a w) g_{c\lambda} - (\partial_\lambda w) g_{ac}]$$

$$= \Gamma^b_{ac} + [(\partial_c w) g^b_a + (\partial_a w) g^b_c - (\partial^b w) g_{ac}]$$

$$\Rightarrow \nabla_a t^b \to \nabla_a t^b + g^b{}_a (\partial_c w) t^c + \cancel{g} (\partial_a w) t^b - (\partial^b w) t_a$$

$$\Rightarrow t^a \nabla_a t^b \to t^a \nabla_a t^b + (\partial_c w) t^c t^b + (\partial_a w) t^a t^b - (\partial^b w) t_a t^a$$

$$= t^a \nabla_a t^b + 2 (\partial_c w) t^c t^b - (\partial^b w) \qquad (t_a t^a = 1)$$

$$\Rightarrow n_b t^a \nabla_a t^b \to n_b t^a \nabla_a t^b \boxed{- (\partial^b w) n_b} \qquad (t^b n_b = 0).$$

Plugging this back into $\frac{1}{2\pi} \int_{\partial M} ds\, k$, $\quad k = t^a n_b \nabla_a t^b$

$$\Delta_{weyl} \frac{1}{2\pi} \int_{\partial M} ds\, k = \frac{1}{2\pi} \int_{\partial M} ds\, (\Delta k)$$

$$= \boxed{\frac{1}{2\pi} \int_{\partial M} ds\, (-\partial^b w) n_b}$$

This matches $\Delta \chi$ for $\chi = \frac{1}{4\pi} \int d^2\sigma^a (-\gamma)^{1/2} R$ we found earlier, except for a sign change, but we can just let ~~[illegible]~~ $k \to \pm k$.

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