Schwartz 5.3(a)

$$\Gamma = \int \frac{1}{2m} |\mathcal{M}| \frac{1}{(2\pi)^5} \frac{d^3\vec{p}_e \, d^3\vec{p}_{\nu_e} \, d^3\vec{p}_{\nu_\mu}}{2^3 |\vec{p}_e||\vec{p}_{\nu_e}||\vec{p}_{\nu_\mu}|} \delta^4(\Sigma p)$$

working in com frame of μ:

$$= \int f(|\vec{p}_{\nu_e}|, m) \frac{d^3\vec{p}_{\nu_e} \, d^3\vec{p}_e}{|\vec{p}_e||\vec{p}_{\nu_e}||\vec{p}_{\nu_e} + \vec{p}_e|} \delta(m - |\vec{p}_e| - |\vec{p}_{\nu_e}| - |\vec{p}_e + \vec{p}_{\nu_e}|)$$

(integrated out $d^3\vec{p}_{\nu_\mu}$ to enforce $-\vec{p}_{\nu_\mu} = \vec{p}_{\nu_e} + \vec{p}_e$.

$$f \equiv \frac{|\mathcal{M}(|\vec{p}_{\nu_e}|, m)|^2}{2m(2\pi)^5 \cdot 8}$$

$$\text{RHS} = \int f(|\vec{p}_{\nu_e}|, m) \underbrace{\frac{|\vec{p}_{\nu_e}||\vec{p}_e| \, d|\vec{p}_{\nu_e}| \, d|\vec{p}_e| \, d\cos\theta_{\nu_e} \, d\cos\theta_e \, d\phi_{\nu_e} \, d\phi_e}{|\vec{p}_{\nu_e} + \vec{p}_e|} \times \delta(m - |\vec{p}_e| - |\vec{p}_{\nu_e}| - |\vec{p}_e + \vec{p}_{\nu_e}|)}$$

only function of $|\vec{p}_{\nu_e}|$, $|\vec{p}_e|$ and the solid angle between. define the coordinate system of $\vec{p}_e$ to have z-axis aligned with $\vec{p}_{\nu_e}$

$$\text{RHS} = \int f(|\vec{p}_{\nu_e}|, m) \frac{|\vec{p}_{\nu_e}||\vec{p}_e| \, d\vec{p}_{\nu_e} \, d\vec{p}_e \, d\cos\theta_{\nu_e} \, d\cos\theta \, d\phi_{\nu_e} \, d\phi_e}{|\vec{p}_{\nu_e} + \vec{p}_e|} \delta(m - |\vec{p}_e| - |\vec{p}_{\nu_e}| - |\vec{p}_e + \vec{p}_{\nu_e}|)$$

these 3 can be integrated, no dependence on them

$$= 2(2\pi)^2 \int f(p, m) \frac{pq \, dp \, dq \, dx}{\sqrt{p^2 + q^2 + 2pqx}} \delta(m - p - q - \sqrt{p^2 + q^2 + 2pqx})$$

$p \equiv |\vec{p}_{\nu_e}|$, $q \equiv |\vec{p}_e|$, $x \equiv \cos\theta$

$$= 2(2\pi)^2 \int f(p, m) \frac{pq \, dp \, dq \, dx}{\sqrt{p^2 + q^2 + 2pqx}} \frac{1}{\left| +1 + \frac{(q + px)}{\sqrt{p^2 + q^2 + 2pqx}} \right|} \delta(q - q_*)$$

Recall $\delta(f(x)) = \sum_{x_i} \frac{1}{\left|\frac{\partial f}{\partial x}\right|} \delta(x - x_i)$,

Here q_* satisfies $m - p - q_* = \sqrt{p^2 + q_*^2 + 2pq_* x}$

$$= 2(2\pi)^2 \int f(p,m)\, \frac{pq\, dp\,dq\,dx}{\cancel{\sqrt{p^2+q^2+2pqx}}} \frac{\cancel{\sqrt{p^2+q^2+2pqx}}}{q+px+\sqrt{p^2+q^2+2pqx}} \delta(q-q_*)$$

Solving for q_*: $(m-p-q_*)^2 = p^2+q_*^2+2pq_*x$

$$m^2+\cancel{p^2}+\cancel{q_*^2}-2mp-2mq_*+2pq_* = \cancel{p^2}+\cancel{q_*^2}+2pq_*x$$

$$m(m-2p) = 2q_*(px+m-p)$$

$$q_* = \frac{m(m-2p)}{2(m+p(x-1))}$$

$q_* \geq 0$ demands $p \leq \frac{m}{2}$.

$$\text{RHS} = 2(2\pi)^2 \int f(p,m)\, \frac{p\, q_*\, dp\, dx}{m+p(x-1)}$$

$$= \cancel{2}(2\pi)^2 \int f(p,m)\, \frac{p\, m(m-2p)}{\cancel{2}[m+p(x-1)]^2}\, dp\, dx$$

$$= (2\pi)^2 \int \frac{32 G_F^2 (m^2-2mp)\, \cancel{m}p^2 (m-2p)}{2\cancel{m}(2\pi)^5\, 8\, [m+p(x-1)]^2}\, dp\, dx$$

$$= \frac{2 G_F^2 m^2}{(2\pi)^3} \int_0^{\frac{m}{2}} \int_{-1}^{1} \frac{(m-2p)^2 p^2}{[m+p(x-1)]^2}\, dx\, dp$$

$$= \frac{4 G_F^2 m^{\cancel{2}}}{(2\pi)^3} \int_0^{\frac{m}{2}} \frac{(m-2p)^2 p^2}{\cancel{m}(m-2p)}\, dp$$

$$= \frac{4 G_F^2 m}{(2\pi)^3} \int_0^{m/2} [mp^2 - 2p^3]\, dp$$

$$= \frac{4 G_F^2 m}{(2\pi)^3} \left[\frac{m}{3}\frac{m^3}{2^3} - \frac{2m^4}{4\cdot 2^4}\right]$$

$$= \frac{4 G_F^2 m^5}{8\pi^3} \frac{1}{96} = \boxed{\frac{G_F^2 m^5}{192\pi^3}}$$

Schwartz 5.3 (b)

Our result: $\Gamma = \frac{G_F^2 m^5}{192\pi^3} = \frac{(1.166)^2 \times 10^{-10}}{GeV^4} \frac{(106)^5 \ MeV^5}{192\pi^3}$

$$= \frac{1.36\times10^{-10} \times 1.338\times10^{10} \times 10^{30} eV^5}{10^{36} eV^4 \ 192 \cdot 3.14}$$

$$= \frac{1.82}{3 \cdot 1895} \times \frac{10^{30}}{10^{36}} eV \approx 3.3 \times 10^{-10} eV$$

$\frac{1}{\Gamma} = 3\times10^{9} eV^{-1}$ we now need to go to "seconds" units.

$[eV^{-1}] = \frac{1}{energy}$ $\hbar = 1.05\times10^{-34} J\cdot s$

$= 6.58\times10^{-16} eV\cdot s$

$$\frac{1}{\Gamma} = \frac{3\times10^9}{\cancel{eV}} \times 6.58\times10^{-16} \cancel{eV}\cdot s = 1.97\times10^{-6} s$$

$$= 1.97 \mu s.$$

$$\frac{2.2-2}{2.2} = \boxed{9\%}$$

There are many factors that can account for this discrepancy

1. ν_e, ν_μ, e masses
2. loop corrections
3. systematic error

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