Schwartz
11.5 (a)

$$i\bar{\psi}\gamma^5\psi = i\psi^{*T}\gamma_0\gamma_5\psi$$

$$C: \; i\psi^{*T}\gamma_0\gamma_5\psi \to i(-i\gamma_2\psi)^T\gamma_0\gamma_5(-i\gamma_2\psi^*)$$

$$= \psi^T\gamma_2^T\gamma_0\gamma_5(-i)\gamma_2\psi^*$$

$$= (-i)\psi^T\gamma_2\gamma_0\gamma_5\gamma_2\psi^*$$

$$\{\gamma_2,\gamma_0\} = 0, \quad \{\gamma_2,\gamma_5\} = 0$$

$$= (-i)\psi^T\gamma_0\gamma_5(\gamma_2\gamma_2)\psi^*$$

$$= i\psi^T\gamma_0\gamma_5\psi^*$$

$$= i(\gamma_0\gamma_5)_{\alpha\beta}\psi_\alpha\psi^*_\beta.$$

In Weyl Rep. $\gamma_0\gamma_5 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} -1 & \\ & 1\end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

Then $(\gamma_0\gamma_5)_{\alpha\beta} = -(\gamma_0\gamma_5)_{\beta\alpha}$,

Assuming ψ is Grassmann, then $\psi_\alpha\psi^*_\beta = -\psi^*_\beta\psi_\alpha$,

So $\quad i(\gamma_0\gamma_5)_{\alpha\beta}\psi_\alpha\psi^*_\beta = i(-\gamma_0\gamma)_{\beta\alpha}(-\psi^*_\beta\psi_\alpha)$

$$= i\psi^\dagger(\gamma_0\gamma_5)\psi$$

$$= \boxed{i\bar{\psi}\gamma_5\psi}$$

(b) $i\bar{\psi}\gamma^5\gamma^\mu\psi = i\psi^{*T}\gamma_0\gamma_5\gamma_\mu\psi$

$C: \psi^* \to -i\gamma_2\psi, \qquad \psi \to -i\gamma_2\psi^*$

$\Rightarrow C: i\psi^{*T}\gamma_0\gamma_5\gamma_\mu\psi$

$\downarrow$

$(i)(-i\gamma_2\psi)^T\gamma_0\gamma_5\gamma_\mu(-i\gamma_2\psi^*)$

$= (-i)\psi^T\gamma_2\gamma_0\gamma_5\gamma_\mu\gamma_2\psi^*$

$= (-i)\psi^T\gamma_0\gamma_5(\gamma_2\gamma_\mu\gamma_2)\psi^*$

$$\gamma_2\gamma_\mu\gamma_2 = \begin{cases} -\gamma_2 & \mu = 2 \\ \gamma_\mu & \mu = 0,1,3. \end{cases}$$

$$= (-i)\psi^T\gamma_0\gamma_5\begin{pmatrix}\gamma_0\\ \gamma_1\\ -\gamma_2\\ \gamma_3\end{pmatrix}\psi^*$$

$$(-i)\, \psi^T \gamma_0 \gamma_5 \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ -\gamma_2 \\ \gamma_3 \end{pmatrix} \psi^*$$

$$= (-i) \left[\gamma_0 \gamma_5 \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ -\gamma_2 \\ \gamma_3 \end{pmatrix} \right]_{\alpha\beta} \psi_\alpha \psi^*_\beta$$

$$= (-i) \left[\begin{pmatrix} \gamma_0^T \\ \gamma_1^T \\ -\gamma_2^T \\ \gamma_3^T \end{pmatrix} \gamma_5^T \gamma_0^T \right]_{\beta\alpha} (-\psi^*_\beta \psi_\alpha)$$

Of course, the Transpose is over the $\alpha\beta$ index

$$= (-i) \left[\begin{pmatrix} \gamma_0 \\ -\gamma_1 \\ -\gamma_2 \\ -\gamma_3 \end{pmatrix} \gamma_5 \gamma_0 \right]_{\beta\alpha} (-\psi^*_\beta) \psi_\alpha$$

$\{\gamma_5, \gamma_\mu\} = 0$, using anticommutators $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$,

$-\gamma_1 \gamma_5 \gamma_0 = -\gamma_1 \gamma_0 \gamma_5 = \gamma_0 \gamma_5 \gamma_1$,

$-\gamma_2 \gamma_5 \gamma_0 = \gamma_0 \gamma_5 \gamma_2$,

$-\gamma_3 \gamma_5 \gamma_0 = \gamma_0 \gamma_5 \gamma_3$, we have

$$(-i)\left[\begin{pmatrix}\gamma_0\\-\gamma_1\\-\gamma_2\\-\gamma_3\end{pmatrix}\gamma_5\gamma_0\right]_{\beta\alpha}(-\psi_\beta^*)\psi_\alpha$$

$$=(-i)\left[\gamma_0\gamma_5\begin{pmatrix}\gamma_0\\\gamma_1\\\gamma_2\\\gamma_3\end{pmatrix}\right]_{\beta\alpha}(-\psi_\beta^*)\psi_\alpha$$

$$= i\,\psi^{*T}\gamma_0\gamma_5\gamma_\mu\psi$$

$$= \boxed{i\,\bar{\psi}\gamma_0\gamma_5\gamma_\mu\psi.}$$

Schwartz 11.5(c)

$$\bar{\psi}\sigma^{\mu\nu}\psi = \psi^{*T}\gamma_0 \frac{i}{2}[\gamma_\mu, \gamma_\nu]\psi$$

$$C: \psi^* \to -i\gamma_2\psi, \qquad \psi \to -i\gamma_2\psi^*$$

$$C: \psi^{*T}\gamma_0 \frac{i}{2}[\gamma_\mu,\gamma_\nu]\psi \to (-i\gamma_2\psi)^T\gamma_0 \frac{i}{2}[\gamma_\mu,\gamma_\nu](-i\gamma_2\psi^*)$$

$$= -\psi^T\gamma_2\gamma_0 \frac{i}{2}[\gamma_\mu,\gamma_\nu]\gamma_2\psi^*$$

$$= \psi^T\gamma_0\gamma_2 \frac{i}{2}[\gamma_\mu,\gamma_\nu]\gamma_2\psi^*$$

If there is no 2 in μ, ν, or $\mu = \nu = 2$, then

$$\gamma_2[\gamma_\mu,\gamma_\nu]\gamma_2 = -[\gamma_\mu,\gamma_\nu] \quad \text{by anticommutation}$$

$$\Rightarrow \quad = \psi^T\gamma_0\left(-\frac{i}{2}\right)[\gamma_\mu,\gamma_\nu]\psi^*$$

$$= -\psi^T\gamma_0\sigma_{\mu\nu}\psi^*$$

$$= +\psi^{*T}(\gamma_0\sigma_{\mu\nu})^T\psi \qquad \text{Recall Grassmann \#s.}$$

$$= \psi^\dagger(\gamma_0\sigma_{\mu\nu})^T\psi$$

$$= \psi^\dagger\sigma_{\mu\nu}^T\gamma_0^T\psi$$

$$= \psi^\dagger\gamma_0(\gamma_0\sigma_{\mu\nu}^T\gamma_0)\psi.$$

Nonzero, independent components of $\sigma_{\mu\nu}$ are

$$\sigma_{12} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad \sigma_{13} = i \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}, \quad \sigma_{23} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$$

$$\sigma_{01} = i \begin{pmatrix} & -1 & & \\ -1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \quad \sigma_{02} = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}, \quad \sigma_{03} = i \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

The ones that contain no 2 in μ, ν, are σ_{13}, σ_{01}, σ_{03}, we quickly compute

$$\gamma_0 \sigma_{13}^T \gamma_0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} a^T & \\ & a^T \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad a = i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = -a^T$$

$$= \begin{pmatrix} a^T & \\ & a^T \end{pmatrix}, \quad = \boxed{-\sigma_{13}}$$

$$\gamma_0 \sigma_{01}^T \gamma_0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} a^T & \\ & -a^T \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad a = i \begin{pmatrix} & -1 \\ -1 & \end{pmatrix} = a^T$$

$$= \begin{pmatrix} -a^T & \\ & a^T \end{pmatrix} = \boxed{-\sigma_{01}}$$

$$\gamma_0 \sigma_{03}^T \gamma_0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} a^T & \\ & -a^T \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad a = i \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} = a^T$$

$$= \boxed{-\sigma_{03}}$$

Thus, for $\mu \neq 2, \nu \neq 2$, we have

$$\psi^\dagger \gamma_0 (\gamma_0 \sigma_{\mu\nu}^T \gamma_0) \psi = \boxed{-\psi^\dagger \gamma_0 \sigma_{\mu\nu} \psi}$$

For there is exactly one of μ, ν equal to 2, we will have

$$C: \bar{\psi} \sigma_{\mu\nu} \psi \to -\psi^\dagger \gamma_0 (\gamma_0 \sigma_{\mu\nu}^T \gamma_0) \psi.$$

We now compute $\gamma_0 \sigma_{\mu\nu}^T \gamma_0$ for exactly one of μ, ν being 2, there are only 3 nonzero, independent ones: $\sigma_{12}, \sigma_{02}, \sigma_{23}$:

$$\gamma_0 \sigma_{12}^T \gamma_0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} a^T & \\ & a^T \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad a = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = a^T$$

$$= \begin{pmatrix} a^T & \\ & a^T \end{pmatrix} = \sigma_{12}$$

$$\gamma_0 \sigma_{02}^T \gamma_0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} a^T & \\ & -a^T \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad a = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -a^T$$

$$= \begin{pmatrix} -a^T & \\ & a^T \end{pmatrix} = \begin{pmatrix} a & \\ & -a \end{pmatrix} = \sigma_{02}$$

$$\gamma_0 \sigma_{23}^T \gamma_0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} a^T & \\ & a^T \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad a = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = a^T$$

$$= \begin{pmatrix} a^T & \\ & a^T \end{pmatrix} = \begin{pmatrix} a & \\ & a \end{pmatrix} = \sigma_{23}.$$

Thus, for there being exactly one 2 in μ, ν,

$$(\gamma_0 \sigma_{\mu\nu}^T \gamma_0) = \sigma_{\mu\nu}, \quad \text{and}$$

$$-\psi^\dagger \gamma_0 (\gamma_0 \sigma_{\mu\nu}^T \gamma_0) \psi$$

$$= -\psi^\dagger \gamma_0 \sigma_{\mu\nu} \psi$$

$$= \boxed{-\bar{\psi} \sigma_{\mu\nu} \psi}$$

Davidson Cheng

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