

9.3: Show that $\text{Tr}(\lambda^\alpha \lambda^\beta) = 2\delta^{\alpha\beta}$

For diagonal terms, it's obvious each λ matrix when squared has trace $\neq 2$.

For the off diagonal terms, first notice that the trace vanishes for product of a symmetric and antisymmetric matrix: suppose $g_{\alpha\beta}$ is symmetric, $f^{\alpha\beta}$ is antisymmetric, then

$$\begin{aligned}\text{Tr}(gf) &= g_{\alpha\beta} f^{\beta\alpha} = -g_{\alpha\beta} f^{\alpha\beta} = -g_{\beta\alpha} f^{\alpha\beta} \\ &\Rightarrow \text{Tr}(gf) = 0.\end{aligned}$$

By explicit computation, one can show that the product of 2 symmetric λ 's and the product of 2 antisymmetric λ 's have their trace vanish unless they are equal:

$$\lambda_1 \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 \lambda_5 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad \lambda_5 \lambda_6 = \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_6 \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_1 \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 \lambda_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 \lambda_8 = \begin{pmatrix} 0 & 0 & 2i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{3}},$$

$$\lambda_1 \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 \lambda_6 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 \lambda_8 = \begin{pmatrix} 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{3}}, \quad \lambda_6 \lambda_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix} \frac{1}{\sqrt{3}},$$

$$\lambda_1 \lambda_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 \lambda_8 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{3}},$$

$$\lambda_1 \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_2 \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 \lambda_7 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_2 \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$