

Griffiths

$$4.23. (a) S_z = \hbar \begin{pmatrix} 1 & \\ & 0 \\ & & -1 \end{pmatrix}.$$

$$(b) S_+ \propto \begin{pmatrix} 0 & a & \\ & 0 & b \\ & & 0 \end{pmatrix}, \quad S_- \propto \begin{pmatrix} 0 & & \\ a & 0 & \\ & b & 0 \end{pmatrix}.$$

$$S_{\pm} |s m\rangle = \hbar \sqrt{s(s+1) - m(m \pm 1)} |s(m \pm 1)\rangle.$$

$$\Rightarrow S_+ |1 -1\rangle = \hbar \sqrt{2 - (-1)(0)} |1 0\rangle \\ = \hbar \sqrt{2} |1 0\rangle.$$

$$S_+ |1 0\rangle = \hbar \sqrt{2} |1 1\rangle.$$

$$S_- |1 1\rangle = \hbar \sqrt{2} |1 0\rangle$$

$$S_- |1 0\rangle = \hbar \sqrt{2} |1 -1\rangle.$$

$$\Rightarrow S_+ = \hbar \sqrt{2} \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \quad S_- = \hbar \sqrt{2} \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & 1 & 0 \end{pmatrix}.$$

$$(c) S_x = \frac{1}{2} (S_+ + S_-) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & 1 & 0 \end{pmatrix}.$$

$$S_y = \frac{1}{2i} (S_+ - S_-) = \frac{\hbar}{i\sqrt{2}} \begin{pmatrix} 0 & 1 & \\ -1 & 0 & 1 \\ & -1 & 0 \end{pmatrix}.$$

(d) letting $|\frac{3}{2} \frac{3}{2}\rangle, |\frac{3}{2} \frac{1}{2}\rangle, |\frac{3}{2} -\frac{1}{2}\rangle, |\frac{3}{2} -\frac{3}{2}\rangle$ have orthonormal basis $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

then $S_z = \frac{\hbar}{2} \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & 3 \end{pmatrix}$.

$$S_+ |\frac{3}{2} -\frac{3}{2}\rangle = \hbar \sqrt{\frac{3}{2}(\frac{5}{2}) - (-\frac{3}{2})(-\frac{1}{2})} |\frac{3}{2} -\frac{1}{2}\rangle$$

$$= \frac{\hbar}{\cancel{2}} \sqrt{\frac{15}{4} - \frac{3}{4}} |\frac{3}{2} -\frac{1}{2}\rangle$$

$$= \frac{\hbar}{\cancel{2}} \sqrt{\frac{12}{4}} |\frac{3}{2} -\frac{1}{2}\rangle.$$

$$S_+ |\frac{3}{2} -\frac{1}{2}\rangle = \hbar \sqrt{\frac{3}{2}(\frac{5}{2}) - (-\frac{1}{2})(\frac{1}{2})} |\frac{3}{2} \frac{1}{2}\rangle$$

$$= \hbar \sqrt{\frac{\cancel{6}}{4}} \sqrt{4} |\frac{3}{2} \frac{1}{2}\rangle.$$

$$S_+ |\frac{3}{2} \frac{1}{2}\rangle = \hbar \sqrt{\frac{3}{2} \frac{5}{2} - (\frac{1}{2})(\frac{3}{2})} |\frac{3}{2} \frac{3}{2}\rangle$$

$$= \hbar \sqrt{\frac{12}{4}} |\frac{3}{2} \frac{3}{2}\rangle.$$

$$S_- \left| \frac{3}{2} \frac{3}{2} \right\rangle = \hbar \sqrt{\frac{15}{4} - \left(\frac{3}{2}\right)\left(\frac{3}{2}\right)} \left| \frac{3}{2} \frac{1}{2} \right\rangle$$

$$= \hbar \sqrt{\frac{12}{4}} \left| \frac{3}{2} \frac{1}{2} \right\rangle.$$

$$S_- \left| \frac{3}{2} \frac{1}{2} \right\rangle = \hbar \sqrt{\frac{15}{4} - \frac{1}{2}\left(-\frac{1}{2}\right)} \left| \frac{3}{2} -\frac{1}{2} \right\rangle$$

$$= \hbar \sqrt{4} \left| \frac{3}{2} -\frac{1}{2} \right\rangle$$

$$S_- \left| \frac{3}{2} -\frac{1}{2} \right\rangle = \hbar \sqrt{\frac{15}{4} - \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)} \left| \frac{3}{2} -\frac{3}{2} \right\rangle$$

$$= \hbar \sqrt{\frac{12}{4}} \left| \frac{3}{2} -\frac{3}{2} \right\rangle.$$

$$\Rightarrow S_+ = \hbar \begin{pmatrix} 0 & \sqrt{3} & & \\ & 0 & 2 & \\ & & 0 & \sqrt{3} \\ & & & 0 \end{pmatrix}.$$

$$S_- = \hbar \begin{pmatrix} 0 & & & \\ \sqrt{3} & 0 & & \\ & 2 & 0 & \\ & & \sqrt{3} & 0 \end{pmatrix}$$

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & & \\ \sqrt{3} & 0 & 2 & \\ & 2 & 0 & \sqrt{3} \\ & & & 0 \end{pmatrix}.$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{2i} \begin{pmatrix} 0 & \sqrt{3} & & \\ -\sqrt{3} & 0 & 2 & \\ & -2 & 0 & \sqrt{3} \\ & & -\sqrt{3} & 0 \end{pmatrix}$$