Townsend bolded exercise after (2.60).

$$r = \left(1+\frac{M}{2\rho}\right)^2 \rho \Rightarrow r = \rho + M + \frac{M^2}{4\rho},$$

$$\frac{dr}{d\rho} = 1 - \frac{M^2}{4\rho^2}$$

$$\Rightarrow dr = \left[1 - \left(\frac{M}{2\rho}\right)^2\right] d\rho$$

$$1 - \frac{2M}{r} = 1 - \frac{2M}{\left(1+\frac{M}{2\rho}\right)^2 \rho} = 1 - \frac{2M}{\rho + M + \frac{M^2}{4\rho}}$$

$$= \frac{\rho - M + \frac{M^2}{4\rho}}{\rho + M + \frac{M^2}{4\rho}}$$

$$= \frac{\rho\left(1 - \frac{M}{2\rho}\right)^2}{\rho\left(1+\frac{M}{2\rho}\right)^2}$$

$$= \frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2}$$

Plugging into Schwarzschild metric $ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$

$$ds^2 = -\frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2} dt^2 + \frac{\left(1+\frac{M}{2\rho}\right)^2}{\left(1-\frac{M}{2\rho}\right)^2}\left[1-\left(\frac{M}{2\rho}\right)^2\right]^2 d\rho^2 + \cancel{\left[1-\left(\frac{M}{2\rho}\right)^2\right]} \left(1+\frac{M}{2\rho}\right)^4 \rho^2 d\Omega^2$$

$$= -\frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2} dt^2 + \left(1+\frac{M}{2\rho}\right)^4 \left[d\rho^2 + \rho^2 d\Omega^2\right]$$

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