Townsend 2.20

$$(\mathcal{L}_k g)_{\mu\nu} = \underline{k^\lambda g_{\mu\nu,\lambda}} + k^\lambda{}_{,\mu} g_{\lambda\nu} + k^\lambda{}_{,\nu} g_{\lambda\mu}$$

$$2 D_\mu k_\nu = 2 \left[k_{\nu,\mu} - \Gamma^\alpha_{\mu\nu} k_\alpha \right]$$

$$= 2 \left[k_{\nu,\mu} - \frac{1}{2} k_\alpha g^{\alpha\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \right]$$

$$= 2 k_{\nu,\mu} - k^\lambda g_{\mu\lambda,\nu} - k^\lambda g_{\nu\lambda,\mu} + \underline{k^\lambda g_{\mu\nu,\lambda}}$$

$$(\mathcal{L}_k g)_{\mu\nu} - 2 D_\mu k_\nu = k^\lambda{}_{,\mu} g_{\lambda\nu} + k^\lambda{}_{,\nu} g_{\lambda\mu} - 2 k_{\nu,\mu} + k^\lambda g_{\mu\lambda,\nu} + k^\lambda g_{\nu\lambda,\mu}$$

$$k^\lambda{}_{,\mu} g_{\lambda\nu} = (k^\lambda g_{\lambda\nu})_{,\mu} - k^\lambda g_{\lambda\nu,\mu} = k_{\nu,\mu} - k^\lambda g_{\lambda\nu,\mu}$$

$$\Rightarrow (\mathcal{L}_k g)_{\mu\nu} - 2 D_\mu k_\nu = k_{\mu,\nu} + k_{\nu,\mu} - 2 k_{\nu,\mu}$$

$$= 0 \quad \text{if} \quad k_{\mu,\nu} + k_{\nu,\mu} - 2 k_{\nu,\mu} = 0.$$

$$\Rightarrow (\mathcal{L}_k g)_{\mu\nu} = 2 D_{(\mu} k_{\nu)} \quad \Leftarrow \quad D_\mu k_\nu \text{ with } \mu,\nu \text{ symmetrized.}$$

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