Townsend (2.18)

$$\dot{x}^\mu \to \dot{x}^\mu - \alpha \dot{\xi}^\mu(x)$$

$$= \dot{x}^\mu - \alpha \dot{x}^\omega \partial_\omega \xi^\mu(x)$$

$$= \dot{x}^\mu - \alpha \dot{x}^\omega D_\omega \xi^\mu(x)$$

⇑

we integrate along proper time λ in the action, in that frame, $\partial \doteq D$.

$$\Rightarrow \dot{x}^\mu \dot{x}^\nu \to \dot{x}^\mu \dot{x}^\nu - \alpha\left[\dot{x}^\mu \dot{x}^\omega D_\omega \xi^\nu + \dot{x}^\nu \dot{x}^\omega D_\omega \xi^\mu\right] + O(\alpha^2)$$

$$= \dot{x}^\mu \dot{x}^\nu - \alpha \dot{x}^\omega \left[\dot{x}^\mu D_\omega \xi^\nu + \dot{x}^\nu D_\omega \xi^\mu\right]$$

$$\dot{x}^\mu \dot{x}^\nu g_{\mu\nu} \to \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} - \alpha \dot{x}^\mu \left[\dot{x}^\omega D_\mu \xi^\nu + \dot{x}^\nu D_\mu \xi^\omega\right] g_{\omega\nu}$$

⇑

swapped μ, ω index.

$$\dot{x}^\omega D_\mu \xi^\nu + \dot{x}^\nu D_\mu \xi^\omega = 2 \dot{x}^{(\nu} D_\mu \xi^{\omega)}$$

$$\left[\dot{x}^\omega D_\mu \xi^\nu + \dot{x}^\nu D_\mu \xi^\omega\right] g_{\omega\nu} = 2 \dot{x}^\nu D_\mu \xi_\nu$$

$$\Rightarrow \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} \to \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} - \alpha \dot{x}^\mu \dot{x}^\nu \left[2 D_\mu \xi_\nu\right]$$

⇑

observe symmetry in μ, ν, implies symmetry in the latter μ, ν

$$= \dot{x}^\mu \dot{x}^\nu - \alpha \dot{x}^\nu \dot{x}^\mu \left[2 D_\nu \xi_\mu\right]$$

$$= \boxed{\dot{x}^\mu \dot{x}^\nu - \alpha \dot{x}^\mu \dot{x}^\nu \left[2 D_{(\mu} \xi_{\nu)}\right]}$$

Danielson

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