

Schubz 7.7

(a) Find as many conserved 4-momentum as possible.

$$(i) \quad ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\text{By } m \frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta} p^\nu p^\alpha$$

it's clear that $g_{\nu\alpha,\beta} = 0$, thus

all 4 of p_μ are conserved.

$$(ii) ds^2 = -(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\Rightarrow g_{\alpha\beta} = \begin{pmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & \frac{1}{1-2M/r} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

We use again $m \frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta} p^\nu p^\alpha$

It's clear that $\left\{ \begin{array}{l} g_{\nu\alpha,\phi} = 0, \\ g_{\nu\alpha,t} = 0 \end{array} \right. \Rightarrow \frac{d}{d\tau} p_\phi = 0$

$\Rightarrow \frac{d}{d\tau} p_t = 0$

$$(iii) \quad ds^2 = - \frac{\Delta - a^2 \sin^2 \theta}{\rho^2} dt^2 - 2a \frac{2Mr \sin^2 \theta}{\rho^2} dt d\phi$$

$$+ \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2} \sin^2 \theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta$$

$$g_{\alpha\beta} = \begin{bmatrix} -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} & 0 & 0 & -2a \frac{2Mr \sin^2 \theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -2a \frac{2Mr \sin^2 \theta}{\rho^2} & 0 & 0 & \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2} \sin^2 \theta \end{bmatrix}$$

The identification of $g_{\nu\alpha, \phi} = 0$ is quick since ϕ does not appear in the metric $\Rightarrow \frac{d}{d\tau} p_\phi = 0$.

Also, t does not appear $\Rightarrow \frac{d}{d\tau} p_0 = 0$.

$$(iv) \quad ds^2 = -dt^2 + R^2(t) \left[(1 - kr^2) dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

$$\Rightarrow g_{\alpha\beta} = \begin{bmatrix} -1 & & & \\ & R^2(t) & & \\ & & (1 - kr^2) & \\ & & & r^2 \begin{bmatrix} 1 & \\ & \sin^2\theta \end{bmatrix} \end{bmatrix}$$

$$\boxed{\frac{d}{dr} p_\phi = 0.}$$

(b) We know Cartesian $\rightarrow$ Spherical gives

$$g_{\text{sp}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \rightarrow g_{\text{sp}} = \begin{bmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2 \theta \end{bmatrix}$$

$\Rightarrow ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ is equivalent to

$$g_{\text{sp}} = \begin{bmatrix} -1 & & \\ & 1 & \\ & & r^2 \\ & & & r^2 \sin^2 \theta \end{bmatrix} \text{ in spherical.}$$

$$\Rightarrow ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (i')$$

In this metric, where the dependence of ds^2 on $d\theta, d\phi$ which is spherical symmetric comes in the form of $\begin{pmatrix} r^2 & \\ & r^2 \sin^2 \theta \end{pmatrix}$.

we make the same identification on (ii) and (iv):

$$(ii): g_{\text{sp}} = \begin{pmatrix} & & \\ & r^2 \begin{pmatrix} 1 & \\ & \sin^2 \theta \end{pmatrix} \end{pmatrix}, \quad (iv): g_{\text{sp}} = \begin{pmatrix} & & \\ & R^2(t) r^2 \begin{pmatrix} 1 & \\ & \sin^2 \theta \end{pmatrix} \end{pmatrix}$$

This indeed implies P_θ is conserved.

(c). $\theta > \pi/2$, $p^\theta = 0$. solve for p^r in m .

$$(i'): g_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2 \sin^2 \theta \end{pmatrix}$$

$$\vec{p} \cdot \vec{p} = -m^2$$

$$\vec{p} \cdot \vec{p} = p^\alpha p^\beta g_{\alpha\beta}$$

$$= -(p^0)^2 + (p^r)^2 + (p^\theta)^2 + (p^\phi)^2$$

$$\therefore -(p^0)^2 + (p^r)^2 + (p^\phi)^2 = -m^2$$

$$(p^r)^2 = -m^2 + (p^0)^2 + (p^\phi)^2$$

$$= E^2 - m^2 + (p^\phi)^2$$

$$\Rightarrow \boxed{p^r = \sqrt{E^2 - m^2 + (p^\phi)^2}}$$

$$(ii): g_{\alpha\beta} = \begin{pmatrix} -(1-2M/r) & & & \\ & (1-2M/r)^{-1} & & \\ & & r^2 & \\ & & & r^2 \sin^2 \theta \end{pmatrix}$$

$$\vec{p} \cdot \vec{p} = -m^2 = p^\alpha p^\beta g_{\alpha\beta}$$

$$= -(1-2\frac{M}{r})(p^0)^2 + \frac{(p^r)^2}{1-2M/r} + r^2 (p^\theta)^2 + r^2 \sin^2 \theta (p^\phi)^2$$

with $p^\theta = 0$, $(p^0)^2 = E^2$, we have

$$\frac{(p^r)^2}{1-2M/r} = -m^2 + E^2(1-2\frac{M}{r}) - r^2 \sin^2 \theta (p^\phi)^2$$

$$p^r = \sqrt{E^2 - (1-2\frac{M}{r})m^2 - (1-2\frac{M}{r})r^2 \sin^2 \theta (p^\phi)^2}$$

$$(iv): -m^2 = -E^2 + R^2(t) \left[\frac{(p^r)^2}{1-kr^2} + r^2 (p^\theta)^2 + r^2 \sin^2 \theta (p^\phi)^2 \right]$$

Multiply by $\frac{1-kr^2}{R^2}$:

$$(p^r)^2 = \left(\frac{1-kr^2}{R^2} \right) [E^2 - m^2] - r^2 \sin^2 \theta (p^\phi)^2$$

$$\Rightarrow p^r = \sqrt{\left(\frac{1-kr^2}{R^2(t)} \right) [E^2 - m^2] - r^2 \sin^2 \theta (p^\phi)^2}$$

(d) when $k=0$, p_t conserved?

$$k=0 \Rightarrow g_{\alpha\beta} = \begin{bmatrix} -1 & & \\ & R^2(t) & \\ & & r^2 \\ & & & R^2 \sin^2 \theta \end{bmatrix}$$

$$(7.29): m \frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta} p^\nu p^\alpha$$

$$\Rightarrow m \frac{dp_r}{d\tau} = \frac{1}{2} g_{\nu\alpha,r} p^\nu p^\alpha$$

$$g_{\nu\alpha,r} \text{ nonzero when } \begin{cases} \nu = \alpha = t, & g_{tt,r} = 2r R^2 \\ \nu = \alpha = \theta, & g_{\theta\theta,r} = 2r R^2 \sin^2 \theta \end{cases}$$

$$\Rightarrow m \frac{dp_r}{d\tau} = \frac{1}{2} \left[g_{tt,r} p^t p^t + g_{\theta\theta,r} p^\theta p^\theta \right]$$

yet $p^t = p^\theta = 0$ by assumption, thus

$$\boxed{p_r \text{ is conserved}}$$

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