Schutz

7.6. The Geodesic is given by $\nabla_{\vec{p}} \vec{p} = 0$.
This is obtained by the affine reparameterization of the particle's trajectory $\vec{U}(\tau)$, where τ is the proper time.

Recall $\nabla_{\vec{U}} \vec{V} = \dfrac{dx^\alpha}{d\tau} V^\beta{}_{;\alpha} = U^\alpha V^\beta{}_{;\alpha}$.

under affine reparameterization $\tau \to \frac{\tau}{m}$,

$$U = \frac{dx^\alpha}{d\tau} \to \frac{dx^\alpha}{d(\tau/m)} = m\frac{dx^\alpha}{d\tau} = m\vec{U} = \vec{p}.$$

So such reparameterization $\tau \to \frac{\tau}{m}$ can be written with the original variable τ with the particle's trajectory $\vec{U}(\tau)$ replaced by $\vec{p}(\tau)$. So the geodesic equation is replaced as.

$p^\alpha V^\beta{}_{;\alpha} = 0$, for $V^\beta = p^\beta$, this is $p^\alpha p^\beta{}_{;\alpha} = 0$.

Schutz 7.6 Show $p^\alpha p^\beta{}_{;\alpha} = 0$ implies $p^\alpha p_{\beta;\alpha} = 0$.

* The simplest proof is that $p^\beta{}_{;\alpha}$ is tensorial, so

$$p_{\beta;\alpha} = g_{\beta\sigma} p^\sigma{}_{;\alpha} \Rightarrow p^\alpha p_{\beta;\alpha} = g_{\beta\sigma} p^\alpha p^\sigma{}_{;\alpha} = 0.$$

* More sophisticated way:

$$p^\alpha p_{\beta;\alpha} = p^\alpha \left[(g_{\beta\sigma} p^\sigma)_{,\alpha} - p_\lambda \Gamma^\lambda_{\beta\alpha} \right]$$

$$= p^\alpha \left[g_{\beta\sigma,\alpha} p^\sigma + g_{\beta\sigma} p^\sigma{}_{,\alpha} - p_\lambda \tfrac{1}{2} g^{\lambda w} (g_{\beta w,\alpha} + g_{\alpha w,\beta} - g_{\alpha\beta,w}) \right]$$

$$= p^\alpha \left[g_{\beta\sigma,\alpha} p^\sigma + g_{\beta\sigma} p^\sigma{}_{,\alpha} - \tfrac{1}{2} p^\sigma (g_{\beta\sigma,\alpha} + g_{\alpha\sigma,\beta} - g_{\beta\alpha,\sigma}) \right]$$

$$= p^\alpha \left[p^\sigma{}_{,\alpha} g_{\beta\sigma} + p^\sigma \tfrac{1}{2} [g_{\beta\sigma,\alpha} + g_{\beta\alpha,\sigma} - g_{\alpha\sigma,\beta}] \right]$$

$$= p^\alpha \left[p^\sigma{}_{,\alpha} g_{\beta\sigma} + p^\sigma \Gamma_{\beta\sigma\alpha} \right]$$

$$= p^\alpha p^\sigma{}_{,\alpha} g_{\beta\sigma} + p^\alpha p^\sigma \Gamma_{\beta\sigma\alpha}$$

$$= g_{\beta\lambda} p^\alpha p^\lambda{}_{,\alpha} + g_{\beta\lambda} \Gamma^\lambda_{\sigma\alpha} p^\alpha p^\sigma$$

$$= g_{\beta\lambda} [p^\alpha][p^\lambda{}_{,\alpha} + p^\sigma \Gamma^\lambda_{\sigma\alpha}]$$

$$= \boxed{g_{\beta\lambda} p^\alpha p^\lambda{}_{;\alpha} = 0}$$

Davidson Cheng
7.1.2024