Schutz 6.33

(a) $(\theta, \phi, \chi)$ are coordinates for the sphere?

It's obviously that $(\theta, \phi, \chi)$ are on the sphere, so it suffices to show $(\theta, \phi, \chi)$ span the sphere. We do so by a series of 1-to-1 mappings.

$x^2 + y^2 + z^2 + w^2 = r^2$ mapped to $a^2 + z^2 + w^2 = r^2$
where $(x, y) \to a(\cos\phi, \sin\phi)$.

Subsequently, $a^2 + z^2 + w^2 = r^2$ mapped to $b^2 + w^2 = r^2$
where $(a, z) \to b(\sin\theta, \cos\theta)$.

Finally, $b^2 + w^2 = r^2$ mapped to $r^2 = r^2$
where $(b, w) \to r(\sin\chi, \cos\chi)$.

Each mapping is 1-1. Thus the entire mapping is 1-1.

(b). $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta \Rightarrow g_{\chi\chi} = \vec{e}_\chi \cdot \vec{e}_\chi$

$$= (\Lambda^\mu{}_\chi \vec{e}_\mu) \cdot (\Lambda^\nu{}_\chi \vec{e}_\nu)$$

$$= \left(\frac{dw}{d\chi}\right)^2 + \left(\frac{dz}{d\chi}\right)^2 + \left(\frac{dy}{d\chi}\right)^2 + \left(\frac{dx}{d\chi}\right)^2$$

$$= (-r\sin\chi)^2 + (r\cos\chi\cos\theta)^2 + (r\cos\chi\sin\theta\cos\phi)^2 + (r\cos\chi\sin\theta\sin\phi)^2$$

$$= r^2(\sin^2\chi + \cos^2\chi\cos^2\theta + \cos^2\chi\sin^2\theta\cos^2\phi + \cos^2\chi\sin^2\theta\sin^2\phi)$$

$$= r^2.$$

$$g_{\theta\theta} = \left(\frac{dw}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2$$

$$= 0 + (-r\sin\chi\sin\theta)^2 + (r\sin\chi\cos\theta\cos\phi)^2 + (r\sin\chi\cos\theta\sin\phi)^2$$

$$= r^2\left[\sin^2\chi\right]$$

$$g_{\phi\phi} = \left(\frac{dw}{d\phi}\right)^2 + \left(\frac{dz}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2 + \left(\frac{dx}{d\phi}\right)^2$$

$$= 0 + 0 + (r\sin\chi\sin\theta\sin\phi)^2 + (r\sin\chi\sin\theta\cos\phi)^2$$

$$= r^2\sin^2\chi\sin^2\theta .$$

~~For off diagonal terms, observe that~~

$$\cancel{g_{\alpha\beta} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\,\vec{e}_{\mu}\cdot\vec{e}_{\nu} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\,g_{\mu\nu}}$$

~~This term only makes diagonal terms of ($\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}$) show up.~~