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Show that covariant differentiation obeys product rule:

$$(V^{\alpha\beta}W_{\beta\gamma})_{;\mu} = V^{\alpha\beta}{}_{;\mu}W_{\beta\gamma} + V^{\alpha\beta}W_{\beta\gamma;\mu}.$$

$$(V^{\alpha\beta}W_{\beta\gamma})_{;\mu} = T^{\alpha}{}_{\gamma;\mu} \qquad (\text{using } T^{\alpha}{}_{\gamma} = V^{\alpha\beta}W_{\beta\gamma})$$

$$= T^{\alpha}{}_{\gamma,\mu} + T^{\sigma}{}_{\gamma}\Gamma^{\alpha}_{\sigma\mu} - T^{\alpha}{}_{\sigma}\Gamma^{\sigma}_{\gamma\mu}$$

$$= (V^{\alpha\beta}W_{\beta\gamma})_{,\mu} + V^{\sigma\beta}W_{\beta\gamma}\Gamma^{\alpha}_{\sigma\mu} - V^{\alpha\beta}W_{\beta\sigma}\Gamma^{\sigma}_{\gamma\mu}$$

$$= V^{\alpha\beta}{}_{,\mu}W_{\beta\gamma} + V^{\alpha\beta}W_{\beta\gamma,\mu} + V^{\sigma\beta}W_{\beta\gamma}\Gamma^{\alpha}_{\sigma\mu} - V^{\alpha\beta}W_{\beta\sigma}\Gamma^{\sigma}_{\gamma\mu}.$$

$$= (V^{\alpha\beta}{}_{,\mu} + V^{\sigma\beta}\Gamma^{\alpha}_{\sigma\mu})W_{\beta\gamma} + V^{\alpha\beta}(W_{\beta\gamma,\mu} - \cancel{\Gamma}\, W_{\beta\sigma}\Gamma^{\sigma}_{\gamma\mu}).$$

Recall that $V^{\alpha\beta}{}_{;\mu} = V^{\alpha\beta}{}_{,\mu} + V^{\alpha\sigma}\Gamma^{\beta}_{\sigma\mu} + V^{\sigma\beta}\Gamma^{\alpha}_{\sigma\mu}$,

$$W_{\beta\gamma;\mu} = W_{\beta\gamma,\mu} - W_{\beta\sigma}\Gamma^{\sigma}_{\gamma\mu} - W_{\sigma\gamma}\Gamma^{\sigma}_{\beta\mu}.$$

So it only remains to show.

$$V^{\alpha\sigma}\Gamma^{\beta}_{\sigma\mu}W_{\beta\gamma} - V^{\alpha\beta}W_{\sigma\gamma}\Gamma^{\sigma}_{\beta\mu} = 0.$$

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Observe that

$$V^{\alpha\sigma} T^{\beta}_{\ \sigma\mu} W_{\beta\gamma} - V^{\alpha\beta} W_{\sigma\gamma} T^{\sigma}_{\ \beta\mu} \quad \text{is}$$

written completely in tensors, so we can apply an index change.

$$\cancel{V^{\alpha\sigma}} \qquad V^{\alpha\beta} = g^{\beta}_{\ \sigma} V^{\alpha\sigma}.$$

$$\Rightarrow \quad V^{\alpha\sigma} T^{\beta}_{\ \sigma\mu} W_{\beta\gamma} - V^{\alpha\beta} W_{\sigma\gamma} T^{\sigma}_{\ \beta\mu}$$

$$= V^{\alpha\sigma} \left[T^{\beta}_{\ \sigma\mu} W_{\beta\gamma} - g^{\beta}_{\ \sigma} W_{\lambda\gamma} T^{\lambda}_{\ \beta\mu} \right].$$

$$= V^{\alpha\sigma} \left[T^{\beta}_{\ \sigma\mu} W_{\beta\gamma} - W_{\lambda\gamma} T^{\lambda}_{\ \sigma\mu} \right].$$

$$= 0.$$