Schutz

6.30. Calculate the Riemann curvature tensor of the cylinder.

We first derive the metric of the cylinder by a cylindrical coordinate transformation then fix r.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} r\cos t \\ r\sin t \\ z \end{pmatrix}. \Rightarrow g_{rr} = \cancel{\left(\frac{dr}{dx}\right)^2 + \frac{d}{}} \; \vec{e}_r \cdot \vec{e}_r.$$

(3) (3) ~~(3)(3+1)/2~~

$$g_{\alpha\bar{\sigma}} = \left(\Lambda^{\beta}_{\bar{\alpha}} \vec{e}_{\beta}\right) \cdot \left(\Lambda^{\mu}_{\bar{\sigma}} \vec{e}_{\mu}\right)$$

$$\Rightarrow g_{rr} = \left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 = 1.$$

$$\Rightarrow g_{\theta\theta} = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = r^2\sin^2\theta + r^2\cos^2\theta$$
$$= r^2$$

$$g_{zz} = 1$$

$$g_{\theta r} = g_{r\theta} = \left(\frac{dx}{dr}\right)\left(\frac{dx}{d\theta}\right) + \left(\frac{dy}{dr}\right)\left(\frac{dy}{dt}\right) + \left(\frac{dz}{dr}\right)\left(\frac{dz}{dt}\right) = \cos\theta(-r\sin\theta) + (\sin\theta)(r\cos\theta)$$
$$= 0.$$

$$g_{rz} = g_{zr} = \left(\frac{dx}{dr}\right)\left(\frac{dx}{dz}\right) + \left(\frac{dy}{dr}\right)\left(\frac{dy}{dz}\right) + \left(\frac{dz}{dt}\right)\left(\frac{dz}{dz}\right) = 0.$$

$$g_{\theta z} = g_{z\theta} = \left(\frac{dx}{d\theta}\right)\left(\frac{dx}{dz}\right) + \left(\frac{dy}{dt}\right)\left(\frac{dy}{dz}\right) + \left(\frac{dz}{dt}\right)\left(\frac{dz}{dz}\right) = 0.$$

We have found all 6 ~~independent~~ independent components of a symmetric matrix g.

$$\Rightarrow g_{\alpha\beta} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & 1 \end{pmatrix}, \qquad g^{\alpha\beta} = \begin{pmatrix} 1 & & \\ & r^{-2} & \\ & & 1 \end{pmatrix}.$$

~~In 3 dim~~ In 3 dimensions The tensor $R_{\alpha\beta\mu\nu}$ shall have at most $\left[\frac{3(3-1)}{2}\right]\left[\frac{3(3-1)}{2}+1\right]\frac{1}{2} = 6$

independent components for R, since we know R must be antisymmetric on the first two and the second two indices, symmetric on the exchange of first two and last two. Just by looking at the metric, it's obvious that R vanishes, if r is a constant, that is, if a specific cylinder is given. Since the metric would then be a constant thus all derivatives of the metric vanishes, making R vanish.

Suppose r is arbitrary, then we are working with Euclidean Geometry, with cylindrical coordinates.