Schutz 6.11 (a)

$$T^{\alpha}{}_{\beta;\nu} = T^{\alpha}{}_{\beta,\nu} + T^{\mu}{}_{\beta}\Gamma^{\alpha}_{\mu\nu} - T^{\alpha}{}_{\mu}\Gamma^{\mu}_{\beta\nu}$$

Replace $T^{\alpha}{}_{\beta}$ with $V^{\alpha}{}_{;\beta} = V^{\alpha}{}_{,\beta} + \Gamma^{\alpha}_{\mu\beta}V^{\mu}$

$$\Rightarrow V^{\alpha}{}_{;\beta\nu} = V^{\alpha}{}_{,\beta\nu} + (\Gamma^{\alpha}_{\mu\beta}V^{\mu})_{,\nu} + (V^{\mu}{}_{,\beta} + \Gamma^{\mu}_{\sigma\beta}V^{\sigma})\Gamma^{\alpha}_{\mu\nu} - (V^{\alpha}{}_{,\mu} + \Gamma^{\alpha}_{\sigma\mu}V^{\sigma})\Gamma^{\mu}_{\beta\nu}$$

$$V^{\alpha}{}_{;\nu\beta} = V^{\alpha}{}_{,\nu\beta} + (\Gamma^{\alpha}_{\mu\nu}V^{\mu})_{,\beta} + (V^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\sigma\nu}V^{\sigma})\Gamma^{\alpha}_{\mu\beta} - (V^{\alpha}{}_{,\mu} + \Gamma^{\alpha}_{\sigma\mu}V^{\sigma})\Gamma^{\mu}_{\beta\nu}.$$

Since $V^{\alpha}{}_{;\beta} = 0$, it's obvious that $V^{\alpha}{}_{;\beta\nu} - V^{\alpha}{}_{;\nu\beta} = 0$.

$$\Rightarrow 0 = V^{\alpha}{}_{;\beta\nu} - V^{\alpha}{}_{;\nu\beta}$$

$$= (\Gamma^{\alpha}_{\mu\beta}V^{\mu})_{,\nu} - (\Gamma^{\alpha}_{\mu\nu}V^{\mu})_{,\beta}$$

$$+ (V^{\mu}{}_{,\beta} + \Gamma^{\mu}_{\sigma\beta}V^{\sigma})\Gamma^{\alpha}_{\mu\nu} - (V^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\sigma\nu}V^{\sigma})\Gamma^{\alpha}_{\mu\beta}.$$

$$= \Gamma^{\alpha}_{\mu\beta,\nu}V^{\mu} + \cancel{\Gamma^{\alpha}_{\mu\beta}V^{\mu}{}_{,\nu}} - \Gamma^{\alpha}_{\mu\nu,\beta}V^{\mu} - \cancel{\Gamma^{\alpha}_{\mu\nu}V^{\mu}{}_{,\beta}}$$

$$+ \cancel{V^{\mu}{}_{,\beta}\Gamma^{\alpha}_{\mu\nu}} + \Gamma^{\mu}_{\sigma\beta}\Gamma^{\alpha}_{\mu\nu}V^{\sigma} - \cancel{V^{\mu}{}_{,\nu}\Gamma^{\alpha}_{\mu\beta}} - \Gamma^{\mu}_{\sigma\nu}\Gamma^{\alpha}_{\mu\beta}V^{\sigma}$$

$$\boxed{= (\Gamma^{\alpha}_{\mu\beta,\nu} - \Gamma^{\alpha}_{\mu\nu,\beta})V^{\mu} - (\Gamma^{\mu}_{\sigma\beta}\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\mu}_{\sigma\nu}\Gamma^{\alpha}_{\mu\beta})V^{\sigma}}$$

(b) trivial

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