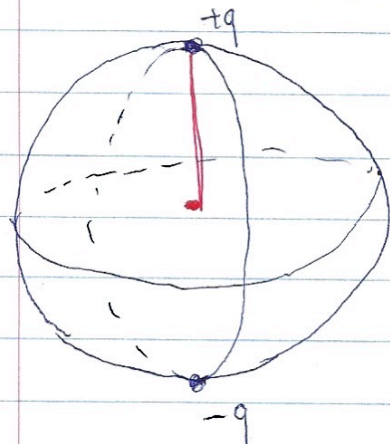
Jackson.

3.6. (a) Inside the sphere, what is the boundary?



This pole between $+q$ and $-q$ is defined by $\theta = 0$, $r \in [0, a]$, gives boundary condition

$$\Phi(r<a, \theta=0, \phi) = kq\left[\frac{1}{a-r} - \frac{1}{a+r}\right]$$

This expression is expanded using geometric series:

$$kq\left[\frac{1}{a-r} - \frac{1}{a+r}\right] = kq\left[\frac{1/a}{\frac{a}{a} - \frac{r}{a}} - \frac{1/a}{\frac{a}{a} + \frac{r}{a}}\right].$$

(putting things in $\frac{r}{a}$ because we are looking for an expansion in $\frac{r}{a}$ since $r<a$)

$$= kq\frac{1}{a}\left[\frac{1}{1-\frac{r}{a}} - \frac{1}{1+\frac{r}{a}}\right]$$

$$= kq\frac{1}{a}\left[\frac{1}{1-\frac{r}{a}} - \frac{1}{1-(-\frac{r}{a})}\right]$$

$$= kq\frac{1}{a}\sum_{i=0}^{\infty}\left[\left(\frac{r}{a}\right)^i - \left(-\frac{r}{a}\right)^i\right].$$

$$= \boxed{\frac{kq(2)}{a}\sum_{i=0}^{\infty}\left(\frac{r}{a}\right)^{2i+1}}$$

This is not the solution, since we are looking for expansion over spherical harmonics.

It's clear that this configuration is ~~spherically~~ azimuthally symmetric, thus expansion in spherical harmonics is equivalent to expansion in Legendre Polynomials:

$$\Phi(r<a, \theta=0, \phi) = \sum_{l=0}^{\infty} A_l r^l P_l[\cos\theta] \Big|_{\theta=0}$$

we recall the simple behavior: $P_l[\cos\theta]\big|_{\theta=0}$ is $P_l[x=1] = 1$.

$$\Rightarrow \Phi(r<a, \theta=0, \phi) = \sum_{l=0}^{\infty} A_l r^l$$

This gives the coefficients $A_l = \frac{2kq}{a}\left(\frac{1}{a}\right)^l$

for odd l, and 0 for all even l:

$$A_l = \begin{cases} \frac{2kq}{a}\left(\frac{1}{a}\right)^l & l \text{ odd} \\ 0 & l \text{ even} \end{cases}$$

~~Clearly, the expansion in~~

This concludes the determination of the Legendre polynomial coefficients using the pole boundary condition, and we write the full expansion over P_l:

$$\boxed{\Phi(r<a, \theta, \phi) = \sum_{l=0}^{\infty} \frac{2kq}{a}\left(\frac{r}{a}\right)^{2l+1} P_l[\cos\theta]}$$

Now for $r > a$,

$$\Phi(r > a, \theta = 0, \phi) = kq\left[\frac{1}{r-a} - \frac{1}{r+a}\right]$$

$$= kq \left[\frac{1/r}{1-\frac{a}{r}} - \frac{1/r}{1+\frac{a}{r}}\right]$$

$$= \frac{kq}{r}\left[\frac{1}{1-\frac{a}{r}} - \frac{1}{1+\frac{a}{r}}\right]$$

(Here, ~~the di~~ the expansion is over $\frac{a}{r}$ since $r > a$)

$$= \frac{kq}{r}\sum_{i=0}^{\infty}\left[\left(\frac{a}{r}\right)^i - \left(-\frac{a}{r}\right)^i\right].$$

$$= \boxed{\frac{2kq}{r}\sum_{i=0}^{\infty}\left(\frac{a}{r}\right)^{2i+1}}$$

This gives B_l since

$$\Phi(r > a, \theta = 0, \phi) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l[\cos\theta]\Big|_{\theta=0}$$

$$= \sum_{l=0}^{\infty} B_l \left(\frac{1}{r}\right)^{l+1}$$

The boundary condition dictates

$$B_l = \begin{cases} 2kq\, a^l & l \text{ odd} \\ 0 & l \text{ even.} \end{cases}$$

Thus we have $\boxed{\Phi(r > a, \theta, \phi) = \sum_{l=0}^{\infty} \cancel{2kq\, a^l r^{-(l+1)}}\; 2kq\, a^{2l+1} r^{-(2l+2)} P_l[\cos\theta]}$

Davidson Cheng

12.30.2023.

Jackson

3.6(b) We previously obtained for $r > a$,

$$\Phi(r>a, \theta, \phi) = \sum_{l=0}^{\infty} 2kq\, a^{2l+1}\, r^{-(2l+2)}\, P_{2l+1}[\cos\theta].$$

take $qa \equiv p/2$ constant,

$$\Phi = kp \sum_{l=0}^{\infty} a^{2l}\, r^{-(2l+2)}\, P_{2l+1}[\cos\theta].$$

$$= kp\left[\frac{1}{r^2} P_1[\cos\theta] + \frac{a^2}{r^4} P_3[\cos\theta] + \cdots\right]$$

$$\cong \boxed{\frac{kp}{r^2}\cos\theta}$$

To see this is the potential of a dipole, expansion of $\Phi(\vec{x})$ for arbitrary configuration is given in Jackson (4.10):

$$\Phi(\vec{x}) = k\left[\frac{q}{r} + \boxed{\frac{\vec{p}\cdot\vec{x}}{r^3}} + \frac{1}{2}\sum_{ij} Q_{ij}\frac{x_i x_j}{r^5} + \cdots\right].$$

$$\Rightarrow \Phi_{dip}(\vec{x}) = k\frac{\vec{p}\cdot\vec{x}}{r^3}$$

Take $\vec{p} = p\hat{z}$, $\vec{p}\cdot\vec{x} = r\cos\theta$.

Davidson Cheng

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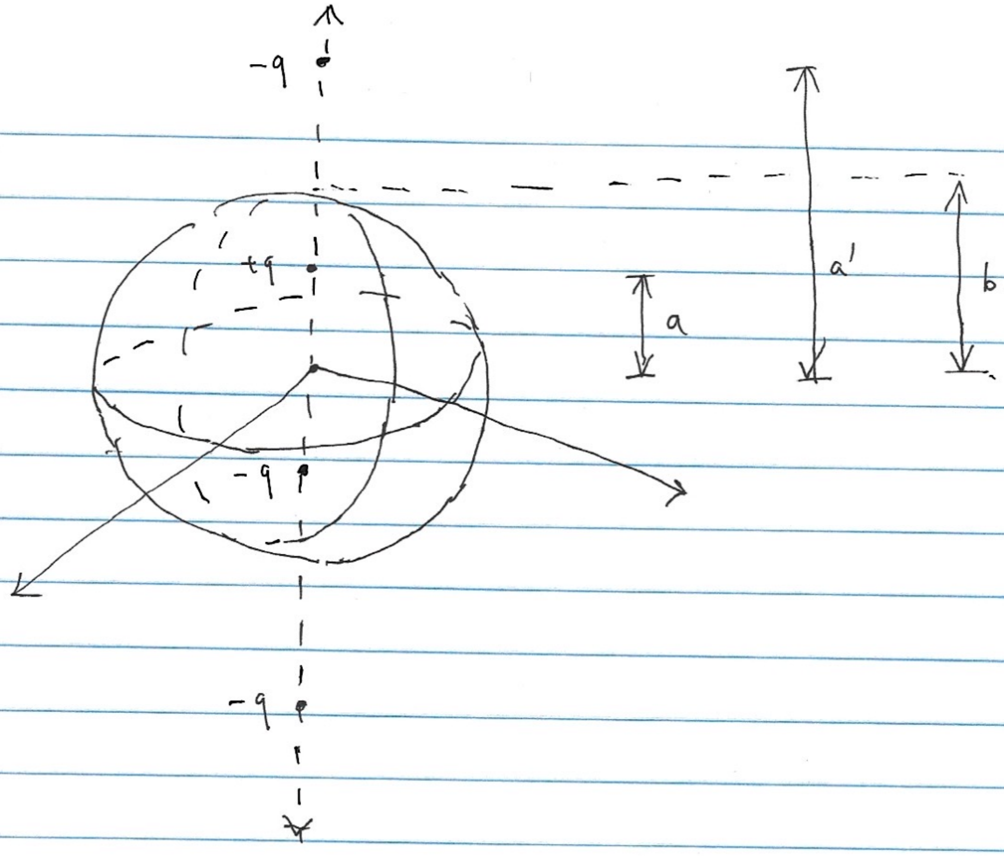
3.6 (c)



Solving the problem of charge q inside grounded conductor via method of images:

$$\Phi(\vec{x}) = \frac{kq}{|x\hat{n} - a\hat{z}|} + \frac{kq'}{|x\hat{n} - a'\hat{z}|}$$

Impose $\Phi(x=b)=0$:

$$0 = \Phi(x=b) = \frac{kq}{|b\hat{n} - a\hat{z}|} + \frac{kq'}{|b\hat{n} - a'\hat{z}|}$$

$$\Rightarrow \quad \frac{kq}{b\,|\hat{n} - \frac{a}{b}\hat{z}|} + \frac{kq'}{a'\,|\hat{z} - \frac{b}{a'}\hat{n}|} = 0.$$

Jackson solved this equation via $\boxed{q' = -\frac{b}{a}q, \quad a' = \frac{b^2}{a}}$

We superimpose the two solutions:

$$\Phi_1 = \frac{kq}{|x\hat{n} - a\hat{z}|} + \left(-\frac{b}{a}\right) q\, k \frac{1}{|x\hat{n} - \frac{b^2}{a}\hat{z}|}$$

The second solution is obtained by $q \to -q$, $\hat{z} \to -\hat{z}$:

$$\Phi_2 = \frac{k(-q)}{|x\hat{n} + a\hat{z}|} + \left(-\frac{b}{a}\right)(-q)\, k \frac{1}{|x\hat{n} + \frac{b^2}{a}\hat{z}|}.$$

$$\Rightarrow \boxed{\Phi = kq \left[|x\hat{n} - a\hat{z}|^{-1} - \left(\frac{b}{a}\right) |x\hat{n} - \frac{b^2}{a}\hat{z}|^{-1} - |x\hat{n} + a\hat{z}|^{-1} + \left(\frac{b}{a}\right) |x\hat{n} + \frac{b^2}{a}\hat{z}|^{-1} \right]}$$

single line, not a vector, (obviously).

This is the contribution from 2 real and 2 image charges.

Davidson Cheng
1-10-2024.