't Hooft String exercise 3.1 pg 12

Observe for $\sigma^{\pm} = \frac{1}{\sqrt{2}}(\sigma \pm \tau)$

$$\sigma = \frac{1}{\sqrt{2}}(\sigma^+ + \sigma^-) \qquad \tau = \frac{1}{\sqrt{2}}(\sigma^+ - \sigma^-)$$

So $X^\mu(\sigma, \tau) = X^\mu(\sigma + \pi, \tau)$

is equivalent to the condition

$$X^\mu(\sigma^+, \sigma^-) = X^\mu\left(\sigma^+ + \frac{\pi}{\sqrt{2}}, \sigma^- + \frac{\pi}{\sqrt{2}}\right)$$

For any closed string, we must have

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + \varphi, \tau) \quad \text{for some } \varphi \neq 0$$

Then $X^\mu(\sigma^+, \sigma^-) = X^\mu\left(\sigma^+ + \frac{\varphi}{\sqrt{2}}, \sigma^- + \frac{\varphi}{\sqrt{2}}\right)$

Then

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+) = \frac{\pi}{\varphi}\sigma^+$$

$$\sigma^- \to \tilde{\sigma}^-(\sigma^-) = \frac{\pi}{\varphi}\sigma^-$$

satisfies $X^\mu(\tilde{\sigma}^+, \tilde{\sigma}^-) = X^\mu\left(\tilde{\sigma}^+ + \frac{\pi}{\sqrt{2}}, \tilde{\sigma}^- + \frac{\pi}{\sqrt{2}}\right)$

Also, in matrix form, $\tilde{\sigma}^{\pm} = \frac{\pi}{\varphi}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\sigma^{\pm}$

$\Rightarrow$ if $h_{\alpha\beta}$ satisfies $h_{\alpha\beta} = \eta_{\alpha\beta} e^{\phi}$,

$$\tilde{h}_{\alpha\beta} = \left[\frac{\pi}{\varphi}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right]^{-1} h_{\alpha\beta} \left[\frac{\pi}{\varphi}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right]$$ does as well.

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