Polchinski A.1 (a)

Let $q(u)$ have expansion

$$Q_0 + \sum_{n=1}^{N} \alpha_n \cos\left[\frac{2\pi n}{U} u\right] + \beta_n \sin\left[\frac{2\pi n}{U} u\right]$$

Then $\dot{q} = \sum_{n=1}^{N} \frac{2\pi n}{U}\left[-\alpha_n \sin\left(\frac{2\pi n}{U} u\right) + \beta_n \cos\left(\frac{2\pi n}{U} u\right)\right]$

We have $S_E = \frac{1}{2}\int_0^U du\, [\dot{q}^2 + \omega^2 q^2]$

$$\dot{q}^2 \cong \sum_{n=1}^{N} \left(\frac{2\pi n}{U}\right)^2 \left[\alpha_n^2 \sin^2\left(\frac{2\pi n}{U} u\right) + \beta_n^2 \cos^2\left(\frac{2\pi n}{U} u\right)\right]$$

$$\omega^2 q^2 \cong \omega^2 Q_0^2 + \omega^2 \sum_{n=1}^{N}\left[\alpha_n^2 \cos^2\left(\frac{2\pi n}{U} u\right) + \beta_n^2 \sin^2\left(\frac{2\pi n}{U} u\right)\right]$$

where terms that would contribute to $\int_0^U du$ are omitted

Then $S_E = \frac{1}{2}\int_0^U du\, [\dot{q}^2 + \omega^2 q^2]$

$$= \frac{1}{2}\sum_{n=1}^{N}\left(\frac{2\pi n}{U}\right)^2 \frac{(\alpha_n^2 + \beta_n^2)}{2} U$$

$$+ \frac{\omega^2}{2} Q_0^2 U + \frac{\omega^2}{2}\frac{U}{2}\sum_{n=1}^{N}(\alpha_n^2 + \beta_n^2)$$

we have used that $\int_0^U du\, \sin^2\left(\frac{2\pi n}{U} u\right) = \int_0^U du\, \cos^2\left(\frac{2\pi n}{U} u\right) = \frac{U}{2}$.

Clean up:

$$S_E = \frac{w^2 Q_0^2 U}{2} + \frac{U}{4}\sum_{n=1}^{N}\left[\left(\frac{2\pi n}{U}\right)^2 + w^2\right](\alpha_n^2 + \beta_n^2).$$

To complete the path integral, integrate over $Q_0, \{\alpha_n\}, \{\beta_n\}$

$$\int dQ_0\, d\alpha_1\, d\alpha_2 \cdots d\alpha_N\, d\beta_1\, d\beta_2 \cdots d\beta_N \times$$

$$\cancel{\exp}\; e^{-\frac{w^2 Q_0^2 U}{2} - \frac{U}{4}\sum_{n=1}^{N}(\alpha_n^2+\beta_n^2)\left[\left(\frac{2\pi n}{U}\right)^2 + w^2\right]}$$

$$= \int dQ_0\, e^{-\frac{w^2 Q_0^2 U}{2}} \int d\alpha_1\, d\alpha_2 \cdots d\alpha_N\, d\beta_1\, d\beta_2 \cdots d\beta_N \prod_{n=1}^{N} e^{-\frac{U}{4}(\alpha_n^2+\beta_n^2)\left[\left(\frac{2\pi n}{U}\right)^2 + w^2\right]}$$

$\#$ $\int dQ_0\, e^{-\frac{w^2 Q_0^2 U}{2}}$ is an gaussian integral, use $\int e^{-x^2} = \sqrt{\pi}$

Let $x = w\sqrt{\frac{U}{2}}\, Q_0$, $dQ_0 = \frac{1}{w}\sqrt{\frac{2}{U}}\, dx$,

$$\int dQ_0\, e^{-\frac{w^2 Q_0^2 U}{2}} = \frac{1}{w}\sqrt{\frac{2}{U}} \int dx\, e^{-x^2} = \frac{1}{w}\sqrt{\frac{2\pi}{U}}.$$

$$\text{Prev. Term} = \sqrt{\frac{2\pi}{U}}\,\frac{1}{w} \int d\alpha_1\, d\alpha_2 \cdots d\alpha_N\, d\beta_1\, d\beta_2 \cdots d\beta_N \prod_{n=1}^{N} e^{-\frac{U}{4}(\alpha_n^2+\beta_n^2)\left[\left(\frac{2\pi n}{U}\right)^2 + w^2\right]}$$

Collecting terms yields, ignore the factor for simplicity for now,

$$\propto \prod_{n=1}^{N} \int d\alpha_n d\beta_n \, e^{-(\alpha_n^2+\beta_n^2)\frac{U}{4}\left[\left(\frac{2\pi n}{U}\right)^2+w^2\right]}$$

$$\int dx\, dy\, e^{-(x^2+y^2)\Lambda}$$

$$= \int r\,dr\,d\theta\, e^{-r^2\Lambda}$$

$$= 2\pi \int_0^{\infty} r\,dr\, e^{-r^2\Lambda}$$

$$= \cancel{2}\pi \frac{e^{-r^2\Lambda}}{-\cancel{2}\Lambda}\Big|_0^{\infty}$$

$$= \frac{\pi}{\Lambda}$$

$$\Rightarrow \propto \prod_{n=1}^{N}\left(\frac{\pi}{\Lambda_n}\right) \qquad \Lambda_n = \frac{U}{4}\left[\left(\frac{2\pi n}{U}\right)^2 + w^2\right]$$

Then we find the entire path integral

$$\sqrt{\frac{2\pi}{U}}\,\frac{1}{\omega}\prod_{n=1}^{\infty}\frac{\pi}{\frac{U}{4}\left[\left(\frac{2\pi n}{U}\right)^2+w^2\right]}$$

$$= \sqrt{\frac{2\pi}{U}} \, \frac{1}{\omega} \prod_{n=1}^{\infty} \frac{\pi U}{\pi^2 n^2 + \frac{U^2 \omega^2}{4}}$$

$$= \sqrt{\frac{U\pi}{2}} \, \boxed{\frac{2}{U\omega} \prod_{n=1}^{\infty} \frac{\pi U}{\pi^2 n^2 + \left(\frac{U\omega}{2}\right)^2}}$$

This term is very sad, because the infinite product representation of $\sinh z = z \prod_{n=1}^{\infty} \left[ 1 + \frac{z^2}{(\pi n)^2} \right]$,

which means $\frac{1}{\sinh \frac{U\omega}{2}}$ would have the infinite product representation

$$\frac{2}{U\omega} \prod_{n=1}^{\infty} \frac{(\pi n)^2}{(\pi n)^2 + \left(\frac{U\omega}{2}\right)^2},$$

Maybe I missed something in my previously computation, but it looks like it can be fixed by going to one of my previous steps and add an "anti-regulator".

Go back to the step of

$$\sqrt{\frac{2\pi}{U}}\frac{1}{\omega}\prod_{n=1}^{N}\int d\alpha_n d\beta_n \, e^{-(\alpha_n^2+\beta_n^2)\frac{U}{4}\left[\left(\frac{2\pi n}{U}\right)^2+\omega^2\right]}$$

Add an "anti-regulator" term "$\frac{\pi n^2}{U}$", we have

$$\sqrt{\frac{2\pi}{U}}\frac{1}{\omega}\prod_{n=1}^{N}\int d\alpha_n d\beta_n \, e^{-(\alpha_n^2+\beta_n^2)\frac{U}{4}\left[\left(\frac{2\pi n}{U}\right)^2+\omega^2\right]} \quad \frac{\pi n^2}{U}$$

Doing the integral in polar coordinates, and carry out the steps that are unaffected by this additional term, we yield

$$(\text{Path integral}) = \sqrt{\frac{2\pi}{U}}\frac{1}{\omega}\prod_{n=1}^{\infty}\frac{\pi^2 n^2/U}{\Lambda_n}, \quad \Lambda_n = \frac{U}{4}\left[\left(\frac{2\pi n}{U}\right)^2+\omega^2\right] \text{ still,}$$

$$= \sqrt{\frac{U\pi}{2}}\frac{2}{U\omega}\prod_{n=1}^{\infty}\frac{\pi^2 n^2}{\pi^2 n^2+\left(\frac{U\omega}{2}\right)^2}$$

$$= \boxed{\sqrt{\frac{U\pi}{2}}\frac{1}{\sinh\left(\frac{U\omega}{2}\right)}}$$

How does this "anti-regulator" term make sense? Because the terms of the infinite product goes to zero too quickly, it is there to make the path integral non-trivial?

$\sum_i \exp(-E_i U)$ is straightforward for Harmonic Oscillator

$$= \sum_{i=0}^{\infty} e^{-(\frac{1}{2}+i)Uw}$$

$$= \sum_{n=0}^{\infty} e^{-\frac{1}{2}Uw} e^{-Uwn}$$

$$= e^{-\frac{1}{2}Uw} \frac{1}{1-e^{-Uw}}$$

$$= \frac{1}{e^{\frac{1}{2}Uw} - e^{-\frac{1}{2}Uw}} \frac{2}{2} = \boxed{\frac{1}{2} \frac{1}{\sinh\left(\frac{Uw}{2}\right)}}$$

Dunlong Chen

8-28-2024