Polchinski 3.6

Begin with $\nabla^a j_a = aR$

under Weyl transformation, $\delta R = -2\nabla^2 \delta w$

$$\Rightarrow \delta_W (aR) = -2a \nabla^2 \delta w$$

$$= -2a \left[g^{z\bar{z}} \nabla_z \nabla_{\bar{z}} \delta w + g^{\bar{z}z} \nabla_{\bar{z}} \nabla_z \delta w \right]$$

In $\begin{bmatrix} z \\ \bar{z} \end{bmatrix}$ coordinates, $g_{ab} = \begin{bmatrix} & 1/2 \\ 1/2 & \end{bmatrix} \Rightarrow g^{ab} = \begin{bmatrix} & 2 \\ 2 & \end{bmatrix}$,

$$= -2a \left[2 \nabla_z \nabla_{\bar{z}} \delta w + 2 \nabla_{\bar{z}} \nabla_z \delta w \right]$$

$$\simeq -8a\, \partial_z \partial_{\bar{z}} \delta w$$

Matching with the left yields $\delta_W [\nabla^a j_a] = -8a\, \partial_z \partial_{\bar{z}} \delta w$

Now, a Weyl transformation $g_{ab} \to e^{2w} g_{ab}$ can be considered as part of a more general transformation $z \to z + \varepsilon v(z)$, in which case $\delta w = \frac{1}{2} [\varepsilon \partial v + \varepsilon \bar{\partial} v^*]$

Plugging this in yields

$$\delta_W [\nabla^a j_a] = -4a\varepsilon \left[\bar{\partial} \partial^2 v + \partial \bar{\partial}^2 v^* \right]$$

The reason we would like to work with $\partial^n v$ and $\bar{\partial}^n v^*$ rather than δw is that it makes it easy to apply Ward's identity in the form of (2.4.11), (2.4.12), as will be seen later.

Now expand the left term $\delta_w[\nabla^a j_a]$

$$\delta_w[\nabla^a j_a] = \delta_w[g^{z\bar z}\nabla_{z} j_{\bar z} + g^{\bar z z}\nabla_{\bar z} j_z]$$

$$= \delta_w[2\nabla_z j_{\bar z} + 2\nabla_{\bar z} j_z]$$

$$\simeq 2[\partial\boxed{\delta_w \tilde{j}} + \bar\partial[\delta_w j]]$$

The equation now reads

$$2\partial[\delta_w \tilde{j}] + 2\bar\partial[\delta_w j] = -4a\varepsilon\,\partial\bar\partial^2 v^* - 4a\varepsilon\,\bar\partial\partial^2 v$$

$$\Rightarrow \quad 2\delta_w \tilde{j} = -4a\varepsilon\,\bar\partial^2 v^*$$

$$2\delta_w j = -4a\varepsilon\,\partial^2 v.$$

Now recall (2.4.11) states that in the expansion $T\mathcal{A}(0)$, given by ~~coefficients~~ terms $\mathcal{A}^{(n)}(0)$

$$T(z)\mathcal{A}(0) \sim \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\mathcal{A}^{(n)}(0),$$

the terms $\mathcal{A}^{(n)}$ will show up in $\delta\mathcal{A}(z,\bar z)$ in a very specific way:

$$\delta\mathcal{A}(z,\bar z) = -\varepsilon\sum_{n=0}^{\infty}\frac{1}{n!}\left[\partial^n v\,\mathcal{A}^{(n)}(z,\bar z) + \bar\partial^n v^*\,\tilde{\mathcal{A}}^{(n)}(z,\bar z)\right]$$

We have derived

$$\delta_w \tilde{j} = -2a\varepsilon\, \partial^2 v^* ,$$

$$\delta_w j = -2a\varepsilon\, \partial^2 v$$

Matching terms, we find $j^{(2)} = +4a$, $\tilde{j}^{(2)} = +4a$.

So putting this back into T_j, $\tilde{T}_{\tilde{j}}$, we see that the ~~sum of the~~ $\frac{1}{z^3}$ term of T_j will be $4a$, and the $\frac{1}{z^3}$ term of $\tilde{T}_{\tilde{j}}$ will be $4a$, and their sum will be $\boxed{8a}$

Davidson's comments: somehow I'm off by a factor of 2. Maybe this came from the convention of the terms in the Laurent expansion? If $\frac{1}{2!}$ is included in $j^{(2)}$ and $\tilde{j}^{(2)}$, then I will no longer be off and get $4a$.

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8-27-2024