# Polchinski 3.3(a)

By defn, covariant derivatives are given by

$$\nabla_a A^b = \partial_a A^b + \Gamma^b_{ac} A^c ,$$
$$\nabla_a A_b = \partial_a A_b - \Gamma^c_{ab} A_c ,$$
$$\nabla_a T^{bc} = \partial_a T^{bc} + \Gamma^b_{\lambda a} T^{\lambda c} + \Gamma^c_{\lambda a} T^{b\lambda}$$

etc.

$$\Gamma_{ab}^{\ \ c} = \frac{1}{2} g^{cd} \left[ g_{ad,b} + g_{bd,a} - g_{ab,d} \right]$$

In conformal gauge, $g_{ab} = \delta_{ab} e^{2w}$

$$\Rightarrow \Gamma_{ab}^{\ \ c} = \cancel{\frac{1}{2}} \cancel{e^{-2w}} \delta^{cd} \left[ \cancel{2} \partial_b w \cancel{e^{2w}} \delta_{ad} + \cancel{2} \partial_a w \cancel{e^{2w}} \delta_{bd} - \cancel{2} \partial_d w \cancel{e^{2w}} \delta_{ab} \right]$$

$$= \delta^{cd} \left[ \partial_b w \, \delta_{ad} + \partial_a w \, \delta_{bd} - \partial_d w \, \delta_{ab} \right]$$

$$= \partial_b w \, \delta_a^{\ c} + \partial_a w \, \delta_b^{\ c} - \partial^c w \, \delta_{ab}$$

Now consider $\nabla_z T^{\bar{z}\bar{z}} = \partial_z T^{\bar{z}\bar{z}} + \Gamma^{\bar{z}}_{cz} T^{c\bar{z}} + \Gamma^{\bar{z}}_{cz} T^{\bar{z}c}$

we would like to compute $\Gamma^{\bar{z}}_{cz}$ to evaluate this:

$$\Gamma^{\bar{z}}_{cz} = \partial_c w \, \delta_z^{\bar{z}} + \partial_z w \, \delta_c^{\bar{z}} - \partial^{\bar{z}} w \, \delta_{cz}$$

In real coordinates, $\gamma_{ab} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, with $z = \sigma^1 + i\sigma^2$, $\bar{z} = \sigma^1 - i\sigma^2$, we would have

$$\gamma_{AB} = \begin{array}{c|c} & A \rightarrow \\ \hline B \downarrow & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}, \quad \text{where } A, B \text{ run over } \{z, \bar{z}\}$$

$$\Rightarrow \gamma_{zz} = \gamma_{\bar{z}\bar{z}} = 0, \quad \gamma_{z\bar{z}} = \gamma_{\bar{z}z} = 1,$$

then $\gamma_z^{\ \bar{z}} = \gamma^{\bar{z}z}\gamma_{zz} + \gamma^{\bar{z}\bar{z}}\gamma_{\bar{z}z} = 0$,

$\gamma_c^{\ \bar{z}} = \gamma^{\bar{z}z}\gamma_{zc} + \gamma^{\bar{z}\bar{z}}\gamma_{\bar{z}c}$, non vanishing when $c = \bar{z}$, $\gamma^{\bar{z}}_{\ \bar{z}} = 1$

$\Rightarrow T_{cz}^{\ \ \bar{z}}$ reduces to $\partial_z w \gamma_c^{\ \bar{z}} - \partial^{\bar{z}} w \gamma_{cz}$, both $\gamma_c^{\ \bar{z}}$ and $\gamma_{cz}$ are non-vanishing only when $c = \bar{z}$, so the only possibly non-vanishing component of $T_{cz}^{\ \ \bar{z}}$ is

$$T_{\bar{z}z}^{\ \ \bar{z}} = \partial_z w - \partial^{\bar{z}} w$$

$$= \partial_z w - \gamma^{\bar{z}z}\partial_z w$$

$$= 0.$$

$\Rightarrow$ All components of $T_{cz}^{\ \ \bar{z}}$ vanish

Now, $\Gamma^{\bar{z}}_{cz}$ is critical in computing covariant derivatives of the form

$$\nabla_z F^{\bar{z}\bar{z}\bar{z}\cdots\bar{z}} = \partial_z F^{\bar{z}\bar{z}\cdots\bar{z}} + \Gamma^{\bar{z}}_{cz} F^{c\bar{z}\cdots\bar{z}} + \Gamma^{\bar{z}}_{cz} F^{\bar{z}c\bar{z}\cdots}$$
$$+ \cdots$$

This implies

$$\boxed{\nabla_z F^{\bar{z}\bar{z}\cdots\bar{z}} = \partial_z F^{\bar{z}\bar{z}\cdots\bar{z}}}$$

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