Polchinski 3.2(a)

We consider n-index traceless symmetric tensor in 2D. The traceless condition states the tensor vanishes when any 2 indices are contracted with $g^{ab} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$.

By symmetry, $T_{a_1 a_2 \cdots a_n}$ is completely fixed by the # of 1s amongst $\{a_1, a_2, \cdots a_n\}$; so we have already eliminated the # of independent components to $n+1$. For illustration, the symmetry condition already leaves a ~~matrix~~ tensor $T_{abc}$ with 4 independent components: $T_{000}$, $T_{100}$, $T_{110}$, $T_{111}$. From now on, we write all 1s to the left.

Now consider the simplest tensor $T_{ab}$, symmetry condition leaves us with 3 components $T_{00}, T_{10}, T_{11}$. Now, we can contract and it gives $T_{00} = -T_{11}$, so we are left with 2 components. We will use "[" to denote a contraction that ~~eliminates~~ combines a set of ~~comp~~ components into 1 independent component:

$$\left[\begin{array}{l} T_{00} \\ T_{10} \\ T_{11} \end{array}\right.$$

For $T_{abcd}$, it can be illustrated with

$$\begin{array}{l} T_{000} \\ T_{100} \\ T_{110} \\ T_{111} \end{array}$$

Again, we are left with 2 independent components by contracting 2 times on 4 components left from symmetry.

For 4-index tensor $T_{abcd}$, it's

$$T_{0000}$$
$$T_{1000}$$
$$T_{1100}$$
$$T_{1110}$$
$$T_{1111}$$

One sees that this procedure continues, when we ~~have~~ add an additional index, it always contracts with the component 2 above it, that is, the component with 2-fewer 1s than it does. So we always have precisely 2 independent components.

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Polchinski 3.2(b)

Let $n$-index traceless $T_{a_1 a_2 \cdots a_n}$ be labeled by indices $\{a_1, a_2, \cdots a_n\}$, we claim the following $(n+1)$-index tensor is traceless symmetric

$$\boxed{\begin{aligned} T_{a_1 a_2 \cdots a_n a_{n+1}} = & \sum_{i=1}^{n+1} \partial_{a_i} T_{a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{n+1}} \\ & - \sum_{\substack{\#\text{ of ways to} \\ \text{pick pair } a_i, a_j \\ \text{from } \{a_1, a_2, \cdots a_{n+1}\}}} \partial_\gamma T^{\gamma}{}_{a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{j-1} a_{j+1} \cdots a_{n+1}} \times g_{a_i a_j} \end{aligned}}$$

e.g. 1 when $n=2$, $T_{ab}$ gets mapped to

$$\partial_a T_{bc} + \partial_b T_{ca} + \partial_c T_{ab}$$
$$- \partial_\gamma T^{\gamma}{}_c g_{ab} - \partial_\gamma T^{\gamma}{}_b g_{ac} - \partial_\gamma T^{\gamma}{}_a g_{bc}$$

e.g. 2. when $n=3$, $T_{abc}$ gets mapped to

$$\partial_a T_{bcd} + \partial_b T_{cda} + \partial_c T_{dab} + \partial_d T_{abc}$$
$$- \partial_\gamma T^{\gamma}{}_{cd} g_{ab} - \partial_\gamma T^{\gamma}{}_{bd} g_{ac} - \partial_\gamma T^{\gamma}{}_{bc} g_{ad}$$
$$- \partial_\gamma T^{\gamma}{}_{ad} g_{bc} - \partial_\gamma T^{\gamma}{}_{ac} g_{bd} - \partial_\gamma T^{\gamma}{}_{ab} g_{cd}.$$

The symmetry follows from the fact that $T_{a_1 a_2 \cdots a_n}$ is symmetric, and both + and − terms in $T_{a_1 a_2 \cdots a_n a_{n+1}}$ ~~are~~ include all permutation of indizes.

The traceless condition can be seen by considering a contraction via $g^{a_i a_j}$ with $T_{a_1 a_2 \cdots a_n a_{n+1}}$, by traceless condition of $T_{a_1 a_2 \cdots a_n}$, we see most terms vanish and we are left only with

$$
\begin{aligned}
g^{a_i a_j} T_{a_1 a_2 \cdots a_n a_{n+1}} = {} & g^{a_i a_j} \partial_{a_i} T_{a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{n+1}} \\
& + g^{a_i a_j} \partial_{a_j} T_{a_1 a_2 \cdots a_{j-1} a_{j+1} \cdots a_{n+1}} \\
& - \partial_\gamma T^{\gamma}{}_{a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{j-1} a_{j+1} \cdots a_{n+1}} \\
& \times g^{a_i a_j} g_{a_i a_j}
\end{aligned}
$$

In the two positive terms, use $g^{a_i a_j}$ to raise $a_i$, $a_j$ respectively, and we see that this ~~term~~ whole term vanish.

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