Polchinski 2.8

First consider $:e^{ik_1X}: \, :e^{ik_2X}: = |z|^{\alpha' k_1 k_2} :e^{i(k_1+k_2)X}:$

$$= z^{\frac{\alpha'}{2}k_1k_2} \bar{z}^{\frac{\alpha'}{2}k_1k_2} :e^{i(k_1+k_2)X}: \qquad (2.2.13)$$

This can be written ~~succinctly~~ alternately as

$$:e^{ik_1X}: \, :e^{ik_2X}:\Big|_{z,\bar{z}} = z^{\frac{\alpha'}{2}k_1k_2} \bar{z}^{\frac{\alpha'}{2}k_1k_2} :e^{i(k_1+k_2)X}:$$

Notice that under $z \to z'$, $\bar{z} \to \bar{z}'$,

$$:e^{ik_1X}: \, :e^{ik_2X}:\Big|_{z',\bar{z}'} = \left(\frac{z'}{z}\right)^{\frac{\alpha'}{2}k_1k_2} \left(\frac{\bar{z}'}{\bar{z}}\right)^{\frac{\alpha'}{2}k_1k_2} \Big[:e^{ik_1X}: \, :e^{ik_2X}:\Big]_{z,\bar{z}}$$

So we deduce $:e^{ikX}:\Big|_{z',\bar{z}'} = \left(\frac{z'}{z}\right)^{\frac{\alpha'}{4}k^2} \left(\frac{\bar{z}'}{\bar{z}}\right)^{\frac{\alpha'}{4}k^2} \Big[:e^{ikX}:\Big]_{z,\bar{z}}$

This implies $:e^{ikX}:$ is not a conformal tensor.

Then

$$\Rightarrow :\partial X \bar{\partial} X e^{ikX}:\Big|_{z',\bar{z}'} = (\partial_z z')(\partial_{\bar{z}} \bar{z}')\left(\frac{z'}{z}\right)^{\frac{\alpha'}{4}k^2} \left(\frac{\bar{z}'}{\bar{z}}\right)^{\frac{\alpha'}{4}k^2} \Big[:e^{ikX}:\Big]_{z,\bar{z}}$$

Recall $\partial X \bar{\partial} X$ has weight $(1,1)$.

Then $\left[f : \partial X \bar{\partial} X e^{ikX} :\right]_{z', \bar{z}'} = (\partial_z z')(\partial_{\bar{z}} \bar{z}') \left(\frac{z'}{z}\right)^{\frac{\alpha'}{4}k^2} \left(\frac{\bar{z}'}{\bar{z}}\right)^{\frac{\alpha'}{4}k^2} A B$
$\times \left[f : \cdots :\right]_{z, \bar{z}}$

where A, B identifies how f transforms under general conformal transformation:

$$f\Big|_{z', \bar{z}'} = A B [f]_{z, \bar{z}}$$

We suppose f has conformal weights $(h_f, \tilde{h}_f)$, this means (as a reminder), for $z' \propto z$, $\bar{z}' \propto \bar{z}$,

$$f\Big|_{z', \bar{z}'} = (\partial_z z')^{h_f} (\partial_{\bar{z}} \bar{z}')^{\tilde{h}_f} [f]_{z, \bar{z}}.$$

$f : \partial X \bar{\partial} X e^{ikX} :$ then has conformal weights $(h_f + \frac{\alpha'}{4} + 1, \tilde{h}_f + \frac{\alpha'}{4} + 1)$

Suppose $f : \partial X \bar{\partial} X e^{ikX} :$ is a conformal tensor, this demands

$$(\partial_z z') \left(\frac{z'}{z}\right)^{\frac{\alpha'}{4}k^2} A = (\partial_z z')^{h_f + \frac{\alpha'}{4}k^2 + 1},$$

$$A = (\partial_z z')^{h_f} \left(\frac{\partial z'}{\partial z} \frac{z}{z'}\right)^{\frac{\alpha'}{4}k^2},$$

Similarly, $B = (\partial_{\bar{z}} \bar{z}')^{\tilde{h}_f} \left(\frac{\partial \bar{z}'}{\partial \bar{z}} \frac{\bar{z}}{\bar{z}'}\right)^{\frac{\alpha'}{4}k^2}$

This gives the conformal trans. property of f under general $z' = z'(z)$, $\bar{z}' = \bar{z}'(\bar{z})$:

$$f|_{z',\bar{z}'} = (\partial_z z')^{h_f} \left(\frac{\partial z'}{\partial z} \frac{z}{z'} \right)^{\frac{\alpha'}{4}k^2} (\partial_{\bar{z}} \bar{z}')^{\tilde{h}_f} \left(\frac{\partial \bar{z}'}{\partial \bar{z}} \frac{\bar{z}}{\bar{z}'} \right)^{\frac{\alpha'}{4}k^2} [f]_{z,\bar{z}}$$

This does not seem trivial as now f has a k dependence in its transformation, it appears that one can write

$$f = \hat{f} : e^{i(ik)X} :$$

where $\hat{f}$ ~~has~~ is a tensor with weights
$(h_f + \frac{\alpha'}{4}k^2, \tilde{h}_f + \frac{\alpha'}{4}k^2)$

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