Polchinski
2.7 (b)

In the Linear Dilaton theory, we are given

$$T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu: + V_\mu \partial^2 X^\mu$$

Consider the OPE $T(z)T(0)$, we don't care about its precise form, but ~~th~~ we know there will be a term of $O(\frac{1}{z^4})$ with no fields at all, ~~...~~

$$T(z)T(0) = \left(-\frac{1}{\alpha'}\right)^2 :\partial X^\mu \partial X_\mu|_z::\partial X^\nu \partial X_\nu|_0: + \underbrace{\cdots}_{\text{we don't care.}}$$

use (2.2.11)

$$\hookrightarrow \quad = \frac{1}{\cancel{\alpha'^2}} \frac{D\cancel{\alpha'^2}}{2 \; z^4} + \cdots$$

Now apply (2.4.20):

$$\mathcal{A}_i(z,\bar z)\mathcal{A}_j(0,0) = \sum_k z^{h_k - h_i - h_j} \bar z^{\tilde h_k - \tilde h_i - \tilde h_j} \mathcal{A}_k(0,0)$$

we see $-h_T - h_T = -4 \Rightarrow h_T = 2$, T has weight $(2,0)$

Now consider OPE $T(z)X^\mu(0)$

$$T(z)X^\mu(0) = -\frac{1}{\alpha'} :\partial X^\nu \partial X_\nu|_z: X^\mu(0) + V_\nu \partial^2 X^\nu(z) X^\mu(0)$$

$$= -\frac{1}{\alpha'} :\partial X^\nu \partial X_\nu|_z::X^\mu|_0: + V_\nu :\partial^2 X^\nu|_z::X^\mu|_0:$$

$$= -\frac{1}{\cancel{\alpha'}} \cancel{2}\left\{-\frac{\cancel{\alpha'}}{\cancel{2}} \partial \ln|z|^2 \eta^{\mu\nu} \partial X_\nu|_z\right\} - V_\nu \frac{\alpha'}{2} \partial^2 \ln|z|^2 \eta^{\mu\nu}$$

$$= \frac{1}{z}\partial X^\mu(0) + O(z) - \frac{1}{z^2}\frac{\alpha'}{2}V^\mu$$

Again, we apply $\mathcal{A}_i \mathcal{A}_j = \sum_k z^{h_k - h_i - h_j} \bar{z}^{\tilde{h}_k - \tilde{h}_i - \tilde{h}_j} \mathcal{A}_k$

~~to find~~ along with T having weight $(2, 0)$,

$\frac{1}{z^2} \frac{\alpha'}{2} \eta^{\mu\nu}$ term tells us X^μ has weight $(0, 0)$

$\frac{1}{z} \partial X^\mu$ term tells us ∂X^μ has weight $(1, 0)$.

By chain rule, ∂X^μ having weight $(1, 0)$ implies that $\partial^2 X^\mu$ has weight $(2, 0)$.

Davidson Cheng
8-26-2024