Polchinski 2.7 (a)

- $X^\mu(z,\bar z) \to X^\mu(z',\bar z')$

$$= X^\mu(z(z'),\bar z(\bar z'))$$

$$= X'^\mu(z',\bar z') \Rightarrow (0,0)$$

- $\partial X^\mu(z,\bar z) \to \partial X^\mu(z',\bar z')$

$$= \left(\frac{\partial z'}{\partial z}\right)\partial_{z'} X^\mu(z',\bar z') \Rightarrow (1,0)$$

- $\bar\partial X^\mu(z,\bar z) \to \bar\partial X^\mu(z',\bar z')$

$$= \left(\frac{\partial \bar z'}{\partial \bar z}\right)\partial_{\bar z'} X^\mu(z',\bar z') \Rightarrow (0,1)$$

- $\partial^2 X^\mu(z,\bar z) \to \partial^2 X^\mu(z',\bar z')$

$$= \partial\left[\left(\frac{\partial z'}{\partial z}\right)\partial_{z'} X^\mu(z',\bar z')\right]$$

$$= \left(\frac{\partial^2 z'}{\partial z^2}\right)\partial_{z'} X^\mu(z',\bar z') + \left(\frac{\partial z'}{\partial z}\right)^2 \partial_{z'} X^\mu(z',\bar z')$$

If $z' = \zeta z$, ζ constant, then $\left(\frac{\partial^2 z'}{\partial z^2}\right)$ vanish and we have a $(2,0)$ weight conformal tensor. Otherwise, if $z' = f(z)$ for general f, we get a 2nd derivative term. ~~this~~ this is not a conformal tensor.

- $:e^{ik\cdot X}:$ we don't plug in $z, \bar{z} \to z', \bar{z}'$ directly, instead consider

$$:e^{ik_1 X}: :e^{ik_2 X}: = |z|^{\alpha' k_1 k_2} :e^{i(k_1+k_2)X}: \qquad (2.2.13) \; (2.2.14)$$

$$= z^{\frac{\alpha'}{2}k_1k_2} \bar{z}^{\frac{\alpha'}{2}k_1k_2} :e^{i(k_1+k_2)X}:$$

under $z \to \zeta z$, $\bar{z} \to \bar{\zeta}\bar{z}$,

$$|z|^{\alpha' k_1 k_2} = z^{\frac{\alpha'}{2}k_1k_2} \bar{z}^{\frac{\alpha'}{2}k_1k_2} \to \zeta^{\frac{\alpha'}{2}k_1k_2} \bar{\zeta}^{\frac{\alpha'}{2}k_1k_2} |z|^{\alpha' k_1 k_2},$$

or more cleanly

$$|z|^{\alpha' k_1 k_2} \to |\zeta|^{\alpha' k_1 k_2} |z|^{\alpha' k_1 k_2}$$

So we must have if we let $h(k)$ denote the conformal weight of e^{ikX}, ~~then~~ ~~and~~ then we must have

$$h(k_1) + h(k_2) = h(k_1+k_2) - \frac{\alpha'}{2}k_1k_2$$

Letting $k_1 = k$, $k_2 = \delta k$,

$$h(k) + h(\delta k) = h(k+\delta k) - \frac{\alpha'}{2} k\,\delta k.$$

$$\frac{h(k+\delta k) - h(k)}{\delta k} = \frac{\alpha'}{2}k + \frac{h(\delta k)}{\delta k}$$

we can write $h(\delta k) = h(0)$ as the δk is infinitesmal.

$$\frac{\partial h(k)}{\partial k} = \frac{\alpha'}{2} k + \frac{h(0)}{\delta k}.$$

I forgot to mention, $h(\delta k)$ must be of order δk^2 or higher for this to make sense, so we can essentially treat it as 0 for δk infinitesmal

$$\frac{\partial h(k)}{\partial k} = \frac{\alpha' k}{2}$$

$$h(k) = \boxed{\frac{\alpha' k^2}{4}}$$

The same argument goes for $\boxed{\tilde{h}(k) = \frac{\alpha' k^2}{4}.}$

This operator is not ~~necessarily~~ a tensor in the above derivation

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8.22.2024