

Polchinski 2.5

Consider infinitesimal $\frac{\delta \mathcal{L}}{\delta \lambda}$, λ some parameter.

$$\frac{\delta \mathcal{L}}{\delta \lambda} = \frac{\delta \mathcal{L}}{\delta \phi_a} \frac{\delta \phi_a}{\delta \lambda} + \frac{\delta \mathcal{L}}{\delta \phi_{a,a}} \frac{\delta \phi_{a,a}}{\delta \lambda}, \quad \text{where } \mathcal{L}(\phi_a, \partial_a \phi_a)$$

~~is given~~ by the Euler-Lagrange Eq. $\partial_a \frac{\delta \mathcal{L}}{\delta \phi_{a,a}} = \frac{\delta \mathcal{L}}{\delta \phi_a}$ follows

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta \lambda} &= \frac{\delta \mathcal{L}}{\delta \phi_a} \frac{\delta \phi_a}{\delta \lambda} + \partial_a \left[\frac{\delta \mathcal{L}}{\delta \phi_{a,a}} \frac{\delta \phi_{a,a}}{\delta \lambda} \right] - \partial_a \frac{\delta \mathcal{L}}{\delta \phi_{a,a}} \frac{\delta \phi_a}{\delta \lambda} \\ &= \partial_a \left[\frac{\delta \mathcal{L}}{\delta \phi_{a,a}} \frac{\delta \phi_{a,a}}{\delta \lambda} \right] \quad \text{with Euler-Lagrange obeyed.} \end{aligned}$$

Letting $\varepsilon = \delta \lambda$, we have

$$\delta \mathcal{L} = \varepsilon \partial_a \left[\frac{\delta \mathcal{L}}{\delta \phi_{a,a}} \frac{\delta \phi_{a,a}}{\delta \lambda} \right]$$

$$\Rightarrow \boxed{K^a = \frac{\delta \mathcal{L}}{\delta \phi_{a,a}} \frac{\delta \phi_{a,a}}{\delta \lambda}}$$

$$\begin{aligned} \text{This further shows } 0 &= \delta \mathcal{L} - \varepsilon \partial_a K^a \\ &= \frac{\delta \mathcal{L}}{\delta \phi_a} \delta \phi_a \varepsilon^{-1} - \partial_a K^a \\ &= \partial_a \left[\frac{\delta \mathcal{L}}{\delta \phi_{a,a}} \right] \delta \phi_a \varepsilon^{-1} - \partial_a K^a \end{aligned}$$

\uparrow
by Euler-Lagrange

$$\Rightarrow \boxed{\partial_a \left[\frac{\delta \mathcal{L}}{\delta \phi_{a,a}} \delta \phi_a \varepsilon^{-1} - K^a \right] = 0 = \partial_a j^a}$$

Davidson Cheng 8.1.2024