Polchinski 2.5

Consider infinitesimal $\frac{\delta \mathcal{L}}{\delta \lambda}$, λ some parameter.

$$\frac{\delta \mathcal{L}}{\delta \lambda} = \frac{\partial \mathcal{L}}{\partial \phi_\alpha} \frac{\delta \phi_\alpha}{\delta \lambda} + \frac{\partial \mathcal{L}}{\partial \phi_{\alpha,a}} \frac{\delta \phi_{\alpha,a}}{\delta \lambda},$$ where $\mathcal{L}(\phi_\alpha, \partial_a \phi_\alpha)$

~~is given~~ follows by the Euler-Lagrange Eq. $\partial_a \frac{\partial \mathcal{L}}{\partial \phi_{\alpha,a}} = \frac{\partial \mathcal{L}}{\partial \phi_\alpha}$.

$$\frac{\delta \mathcal{L}}{\delta \lambda} = \frac{\partial \mathcal{L}}{\partial \phi_\alpha} \frac{\delta \phi_\alpha}{\delta \lambda} + \partial_a \left[\frac{\partial \mathcal{L}}{\partial \phi_{\alpha,a}} \frac{\delta \phi_\alpha}{\delta \lambda} \right] - \partial_a \frac{\partial \mathcal{L}}{\partial (\phi_{\alpha,a})} \frac{\delta \phi_\alpha}{\delta \lambda}.$$

$$= \partial_a \left[\frac{\partial \mathcal{L}}{\partial \phi_{\alpha,a}} \frac{\delta \phi_\alpha}{\delta \lambda} \right]$$ with Euler-Lagrange obeyed.

Letting $\varepsilon = \delta \lambda$, we have

$$\delta \mathcal{L} = \varepsilon \, \partial_a \left[\frac{\partial \mathcal{L}}{\partial \phi_{\alpha,a}} \frac{\delta \phi_\alpha}{\delta \lambda} \right]$$

$$\Rightarrow \boxed{K^a = \frac{\partial \mathcal{L}}{\partial \phi_{\alpha,a}} \frac{\partial \phi_\alpha}{\partial \lambda}}$$

This further shows $$0 = \delta \mathcal{L} - \varepsilon \, \partial_a K^a$$

$$= \frac{\partial \mathcal{L}}{\partial \phi_\alpha} \delta \phi_\alpha \varepsilon^{-1} - \partial_a K^a$$

$$= \partial_a \left[\frac{\partial \mathcal{L}}{\partial \phi_{\alpha,a}} \right] \delta \phi_\alpha \varepsilon^{-1} - \partial_a K^a$$

$\Uparrow$ by Euler Lagrange

$$\Rightarrow \boxed{\partial_a \left[\frac{\partial \mathcal{L}}{\partial \phi_{\alpha,a}} \delta \phi_\alpha \varepsilon^{-1} - K^a \right] = 0 = \partial_a j^a}$$

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