Polchinski 2.9

Consider $e^{ik_1\cdot X(z,\bar z)}$, expanding in $z, \bar z$:

$$e^{ik_1\cdot X(z,\bar z)} = e^{ik_1\cdot X(0,0)} + z\, ik_1^\mu \partial X^\mu|_0\, e^{ik_1\cdot X(z,\bar z)}|_0$$

$$+ \bar z\, i k_1^\mu \bar\partial X^\mu|_0\, e^{ik_1\cdot X(0,0)}$$

$$+ \frac{z^2}{2}\left[i k_1^\mu \partial^2 X^\mu|_0\, e^{ik_1\cdot X(0,0)} - k_1^\mu k_1^\nu \partial X^\mu \partial X^\nu e^{ik_1\cdot X(0,0)}\right]$$

$$+ \frac{\bar z^2}{2}\left[i k_1^\mu \bar\partial^2 X^\mu|_0\, e^{ik_1\cdot X(0,0)} - k_1^\mu k_1^\nu \bar\partial X^\mu \bar\partial X^\nu e^{ik_1\cdot X(0,0)}\right]$$

$$+ \frac{z\bar z}{2}\left[i k_1^\mu \partial\bar\partial X^\mu|_0\, e^{ik_1\cdot X(0,0)} - k_1^\mu k_1^\nu \partial X^\mu \bar\partial X^\nu e^{ik_1\cdot X(0,0)}\right]$$

$$+ O(z^3)$$

$$= e^{ik_1\cdot X(0,0)}\Big\{ 1 + z\, ik_1^\mu \partial X^\mu|_0 + \bar z\, ik_1^\mu \bar\partial X^\mu|_0$$

$$+ \frac{z^2}{2}\left[ik_1^\mu \partial^2 X^\mu|_0 - k_1^\mu k_1^\nu \partial X^\mu \partial X^\nu\right]$$

$$+ \frac{\bar z^2}{2}\left[ik_1^\mu \bar\partial^2 X^\mu|_0 - k_1^\mu k_1^\nu \bar\partial X^\mu \bar\partial X^\nu\right]$$

$$+ \frac{z\bar z^2}{2}\left[ik_1^\mu \partial\bar\partial X^\mu|_0 - k_1^\mu k_1^\nu \partial X^\mu \bar\partial X^\nu\right]$$

$$+ O(z^3)\Big\}$$

Plugging this into the RHS of (2.2.14) we get

$$|z|^{\alpha' k_1\cdot k_2} :e^{i(k_1+k_2)\cdot X(0,0)}\Big\{1 + z\, i k_1^\mu \partial X^\mu|_0 + \bar{z}\, i k_1^\mu \bar\partial X^\mu|_0$$
$$+ z^2(\cdots) + \bar{z}^2(\cdots) + z\bar{z}(\cdots) + O(z^3)\Big\}$$

$$= |z|^{\alpha' k_1\cdot k_2}\Big\{ :e^{i(k_1+k_2)\cdot X(0,0)}:$$

$$+ z\, i k_1^\mu :\partial X^\mu|_0\, e^{i(k_1+k_2)\cdot X(0,0)}:$$

$$+ \bar{z}\, i k_1^\mu :\bar\partial X^\mu|_0\, e^{i(k_1+k_2)\cdot X(0,0)}:$$

$$+ \frac{z^2}{2} : i k_1^\mu \partial^2 X^\mu|_0 - k_1^\mu k_1^\nu \partial X^\mu \partial X^\nu|_0 :$$

$$+ \frac{\bar{z}^2}{2} : i k_1^\mu \bar\partial^2 X^\mu|_0 - k_1^\mu k_1^\nu \bar\partial X^\mu \bar\partial X^\nu|_0 :$$

$$+ \frac{z\bar{z}}{2} : i k_1^\mu \partial\bar\partial X^\mu|_0 - k_1^\mu k_1^\nu \partial X^\mu \bar\partial X^\nu|_0 :$$

$$+ O(z^3)\Big\}$$

To see the ~~conformal~~ $z, \bar{z}$ exponents of $|z|^{\alpha' k_1 \cdot k_2}$, observe

$$|z|^{\alpha' k_1 \cdot k_2} = |z|^{\alpha' \frac{1}{2}[(k_1+k_2)^2 - k_1^2 - k_2^2]}$$

$$\cancel{= [z\bar{z}]^{\frac{1}{2}\alpha'}} \quad \text{recall } |z| = (z\bar{z})^{1/2}$$

$$= (z\bar{z})^{\frac{1}{4}\alpha'[(k_1+k_2)^2 - k_1^2 - k_2^2]}$$

$$= z^{\frac{1}{4}\alpha'[(k_1+k_2)^2 - k_1^2 - k_2^2]} \bar{z}^{\frac{1}{4}\alpha'[(k_1+k_2)^2 - k_1^2 - k_2^2]}$$

Quote (7.26) in Joe's little book,

For $:e^{ik\cdot X}:$ $\quad h(k) = \tilde{h}(k) = \frac{\alpha'}{4} k\cdot k$.

$\Rightarrow$ Inside $\underset{A_i}{\underset{\Uparrow}{:e^{ik_1 X}:}} \underset{A_j}{\underset{\Uparrow}{:e^{ik_2 X}:}}$, $\quad h_i = \tilde{h}_i = \frac{\alpha'}{4} k_1^2$

$h_j = \tilde{h}_j = \frac{\alpha'}{4} k_2^2$

Inside $\underset{A_k}{\underset{\Uparrow}{:e^{i(k_1+k_2)X}:}}$, $\quad h_k = \tilde{h}_k = \frac{\alpha'}{4}(k_1+k_2)^2$.

So In expansion, these terms would give

$$h_k - h_i - h_j = \tilde{h}_k - \tilde{h}_i - \tilde{h}_j = \frac{\alpha'}{4}[(k_1-k_2)^2 - k_1^2 - k_2^2].$$

Adding a ∂X derivative term in A_k gives an $\overbrace{h_k}^{\text{additional}}$ conformal weight, adding a $\bar{\partial} X$ derivative term would give an additional h^2_k conformal factor.

Comparing this with the form of (2.4.20), we see that they agree.

Davidson Chang

8.18.2024.