Polchinski 1.4a

From (1.3.36), we have the mass shell relation

$$m^2 = 2p^+H - p^ip^i = \frac{1}{\alpha'}\left[\sum_{n=1}^{\infty}\sum_{i=2}^{D-1} n N_{ni} - 1\right], \quad D=26 \text{ assumed.}$$

For $m^2 = \frac{1}{\alpha'}$, we have $\sum_{n=1}^{\infty}\sum_{i=2}^{\infty} n N_{ni} = 2$, which

can be achieved with

$$\begin{cases} \alpha_{-1}^i \alpha_{-1}^j |0,k\rangle & i \neq j,\ i,j \in \{2,3,\cdots,25\} \\ \alpha_{-2}^i |0,k\rangle & i \in \{2,3,\cdots,25\} \end{cases}$$

We know both cases has states transforming under a representation of some subgroup of SO(25), we wish to show that together, they transform under a rep. of precisely SO(25)

First consider $\alpha_{-1}^i \alpha_{-1}^j |0,k\rangle$, $[\alpha_{-1}^i, \alpha_{-1}^j] = 0$ so this state must be symmetric in i,j. This symmetric condition together with the $i \neq j$ condition gives us $\frac{(24)(23)}{2} = \frac{(D-2)(D-3)}{2}$

independent states. So we have a $\frac{(24)(23)}{2}$ dimensional rep.

Second consider $\alpha_{-2}^i|0,k\rangle$. This rep. is obviously $D-2 = 24$ dimensional because we have 24 independent states.

Now, if the $\alpha^i_{-1}\alpha^j_{-1}|0,k\rangle$ rep. and $\alpha^i_{-2}|0,k\rangle$ rep. don't mix, we would have in total

$$\frac{(24)(23)}{2}+24=\frac{(D-2)(D-3)}{2}+(D-2)=\frac{(D-1)(D-2)}{2}$$

dimensions. Since $SO(D-1)$ only has $\frac{(D-1)(D-2)}{2}$ dimensions, this rep. must be that of $SO(D-1)$.

To show that $\alpha^i_{-1}\alpha^j_{-1}|0,k\rangle$ and $\alpha^i_{-2}|0,k\rangle$ do not mix, observe that under the harmonic oscillator algebra,

$$\alpha^i_{-2}|0,k\rangle \propto \alpha^i_{-1}\alpha^i_{-1}|0,k\rangle$$

which would have no chance of being in $\alpha^i_{-1}\alpha^j_{-1}|0,k\rangle$ because the $\alpha^i_{-1}\alpha^j_{-1}|0,k\rangle$ states by assumption has $i\neq j$, so they do not mix and we have them together forming a rep. of $SO(D-1)$.

Pandson Chang.

8.17.2024