$$h_{ab} = \partial_a X^\mu \partial_b X_\mu$$

Polchinski 1.1 b

$$X^\mu = (\tau, x, y, z)$$

$$X_\mu = (-\tau, x, y, z)$$

$$\partial_\tau X^\mu = (1, v_x, v_y, v_z) = (1, \dot{x}, \dot{y}, \dot{z})$$

$$\partial_\tau X_\mu = (-1, \dot{x}, \dot{y}, \dot{z})$$

$$\partial_\sigma X^\mu = \left(0, \frac{\partial x}{\partial \sigma}, \frac{\partial y}{\partial \sigma}, \frac{\partial z}{\partial \sigma}\right)$$

$$= (0, x', y', z')$$

$$\partial_\sigma X_\mu = (0, x', y', z')$$

$$h_{\tau\tau} = |\vec{v}|^2 - 1 = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 1)$$

$$h_{\sigma\sigma} = (x'^2 + y'^2 + z'^2)$$

$$h_{\tau\sigma} = (\dot{x}x' + \dot{y}y' + \dot{z}z')$$

$$-\det h_{ab} = [h_{\tau\sigma} h_{\sigma\tau} - h_{\tau\tau} h_{\sigma\sigma}]$$

$$= (\dot{x}x' + \dot{y}y' + \dot{z}z')^2 - (|\vec{v}|^2 - 1)(x'^2 + y'^2 + z'^2)$$

$$= (\dot{x}x' + \dot{y}y' + \dot{z}z'^2) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 1)(x'^2 + y'^2 + z'^2)$$

$$\sqrt{-\det h_{ab}} = \left[(\dot{x}x' + \dot{y}y' + \dot{z}z')^2 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 1)(x'^2 + y'^2 + z'^2)\right]^{1/2}$$

$$\Big|_{\vec{V}=0} = \left[x'^2 + y'^2 + z'^2\right]^{1/2}$$

$$\frac{\partial}{\partial V_x} = \frac{\partial}{\partial \dot{x}} = \frac{1}{2}[\cdots]^{-1/2}\Big[2(\dot{x}x' + \dot{y}y' + \dot{z}z')x' - 2\dot{x}(x'^2 + y'^2 + z'^2)\Big]$$

$$= [\cdots]^{-1/2}\Big[(\dot{x}x' + \dot{y}y' + \dot{z}z')x' - \dot{x}(x'^2 + y'^2 + z'^2)\Big]$$

$$\Big|_{\vec{V}=0} = 0$$

$$\frac{\partial^2}{\partial \dot{x}^2} = \left(-\frac{1}{2}\right)[\cdots]^{-3/2}\Big[2(\dot{x}x' + \dot{y}y' + \dot{z}z')x' - 2\dot{x}(x'^2 + y'^2 + z'^2)\Big]\Big[(\dot{x}x' + \dot{y}y' + \dot{z}z')x' - \dot{x}(x'^2 + y'^2 + z'^2)\Big]$$

$$+ [\cdots]^{-1/2}\Big[x'^2 - (x'^2 + y'^2 + z'^2)\Big]$$

$$= -[\cdots]^{-3/2}\Big[(\dot{x}x' + \dot{y}y' + \dot{z}z')x' - \dot{x}(x'^2 + y'^2 + z'^2)\Big]^2$$

$$+ [\cdots]^{-1/2}\Big[-(y'^2 + z'^2)\Big]$$

$$\Big|_{\vec{V}=0} = \frac{-[y'^2 + z'^2]}{\sqrt{x'^2 + y'^2 + z'^2}}$$

$$\frac{\partial^2}{\partial \dot{y}\,\partial \dot{x}} = -\frac{1}{2}[\cdots]^{-3/2}\left[2(\dot{x}x' + \dot{y}y' + \dot{z}z')y' - 2\dot{y}(x'^2 + y'^2 + z'^2)\right] + [\cdots]^{-1/2}[x'y']$$

$$\Big|_{\vec{v}=0} = \frac{x'y'}{\sqrt{x'^2 + y'^2 + z'^2}}$$

$$\Rightarrow \sqrt{-\det h_{ab}} = (x'^2 + y'^2 + z'^2)^{1/2}$$

$$+ \frac{1}{2}[x'^2 + y'^2 + z'^2]^{-1/2}\left\{\begin{array}{l} -[y'^2 + z'^2]\dot{x}^2 \\ -[z'^2 + x'^2]\dot{y}^2 \\ -[x'^2 + y'^2]\dot{z}^2 \\ +2x'y'\dot{x}\dot{y} \\ +2x'z'\dot{x}\dot{z} \\ +2y'z'\dot{y}\dot{z} \end{array}\right\}$$

The leading term $(x'^2 + y'^2 + z'^2)^{1/2}$ gives

$$S_{NG} = \int d\tau\, d\sigma \left(-\frac{1}{2\pi\alpha'}\right)\sqrt{\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2} + O(v^2)$$

$$= -\frac{1}{2\pi\alpha'}\int d\tau \boxed{\int d\sigma \sqrt{x'^2 + y'^2 + z'^2}}$$

This term corresponds to the length of the string.

The kinetic terms are

$$-\frac{1}{2}\left[x'^2+y'^2+z'^2\right]^{-1/2}\left\{\begin{array}{l}[y'^2+z'^2]\,\dot{x}^2\\ +[z'^2+x'^2]\,\dot{y}^2\\ +[x'^2+y'^2]\,\dot{z}^2\end{array}\right\}.$$

$\dfrac{(y'^2+z'^2)}{\sqrt{x'^2+y'^2+z'^2}}$ is dimensionless and is in the direction of y, z, so

$\displaystyle\int \frac{(y'^2+z'^2)}{\sqrt{x'^2+y'^2+z'^2}}\,d\sigma$ corresponds to infinitesimal length element in y, z plane,

Thus $\frac{1}{2}\left[x'^2+y'^2+z'^2\right]^{-1/2}\left[y'^2+z'^2\right]\dot{x}^2$

is a transverse kinetic term.

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