Polchinski Big Book 1.1 (a)

$$S_{pp} = -m\int d\tau\,(-\dot{x}^\mu \dot{x}_\mu)^{1/2}$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\dot{x}^\mu = (-1, v_x, v_y, v_z)$$

$$\dot{x}_\mu = (1, v_x, v_y, v_z)$$

$$(-\dot{x}^\mu \dot{x}_\mu)^{1/2} = (1 - |\vec{v}|^2)^{1/2}$$

Now expand $(1-|\vec{v}|^2)^{1/2}$ in $|\vec{v}|$, since in non-relativistic limit $|\vec{v}| \ll 1$.

$$(1-|\vec{v}|^2)^{1/2}\Big|_{|\vec{v}|=0} = \boxed{1}$$

$$\frac{\partial}{\partial|\vec{v}|}(1-|\vec{v}|^2)^{1/2} = \frac{1}{\cancel{2}}(1-|\vec{v}|^2)^{-1/2}(-\cancel{2}|\vec{v}|)$$

$$= -(1-|\vec{v}|^2)^{-1/2}|\vec{v}| \qquad \Big|_{|\vec{v}|=0} = \boxed{0}$$

$$\frac{\partial^2}{\partial|\vec{v}|^2}(1-|\vec{v}|^2)^{1/2} = -(\cancel{-}\tfrac{1}{\cancel{2}})(1-|\vec{v}|^2)^{-3/2}(\cancel{-2}|\vec{v}|)|\vec{v}| - (1-|\vec{v}|^2)^{-1/2}$$

$$= -(1-|\vec{v}|^2)^{-3/2}|\vec{v}|^2 - (1-|\vec{v}|^2)^{-1/2} \qquad \Big|_{|\vec{v}|=0} = \underline{-1}$$

$$\Rightarrow (1-|\vec{v}|^2)^{1/2} = 1 - \frac{1}{2}|\vec{v}|^2 + O(|\vec{v}|^3)$$

$$\Rightarrow S_{pp} \approx -m\int d\tau\left[1 - \frac{1}{2}|\vec{v}|^2\right]$$

$$\boxed{= \int d\tau\left[\frac{1}{2}mv^2 - m\right]}$$

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