Polchinski 10.2 (a)

$$\delta_{\eta_2} X = \varepsilon \sqrt{\frac{\alpha'}{2}} \left[\eta_2 \psi + \eta_2^* \tilde{\psi} \right]$$

$$\delta_{\eta_1} \delta_{\eta_2} X = \varepsilon \sqrt{\frac{\alpha'}{2}} \left[\eta_2 (\delta_{\eta_1} \psi) + \eta_2^* (\delta_{\eta_1} \tilde{\psi}) \right]$$

$$\eta_2 (\delta_{\eta_1} \psi) = \eta_2 \left[- \varepsilon \sqrt{\frac{2}{\alpha'}} \eta_1 \partial X \right]$$

$$\eta_2^* (\delta_{\eta_1} \psi) = \eta_2^* \left[- \varepsilon \sqrt{\frac{2}{\alpha'}} \eta_1^* \bar{\partial} X \right].$$

$$\Rightarrow \delta_{\eta_1} \delta_{\eta_2} X = \left(\varepsilon \sqrt{\frac{\alpha'}{2}} \right) \left(- \varepsilon \sqrt{\frac{2}{\alpha'}} \right) \left[\eta_2 \eta_1 \partial X + \eta_2^* \eta_1^* \bar{\partial} X \right]$$

$$= - \varepsilon^2 \left[\eta_2 \eta_1 \partial X + \eta_2^* \eta_1^* \bar{\partial} X \right]$$

$$\Rightarrow \delta_{\eta_2} \delta_{1} X = - \varepsilon^2 \left[\eta_1 \eta_2 \partial X + \eta_1^* \eta_2^* \bar{\partial} X \right]$$

$$\Rightarrow \delta_{\eta_1} \delta_{\eta_2} X - \delta_{\eta_2} \delta_{\eta_1} X = - \varepsilon^2 \left[(\eta_2 \eta_1 - \eta_1 \eta_2) \partial X + (\eta_2^* \eta_1^* - \eta_1^* \eta_2^*) \bar{\partial} X \right]$$

$$(\delta_{\eta_1} \delta_{\eta_2} - \delta_{\eta_2} \delta_{\eta_1}) X = - \varepsilon^2 \left[-2 \eta_1 \eta_2 \partial X - 2 \eta_1^* \eta_2^* \bar{\partial} X \right]$$

using anticommutativity of η_1, η_2.

Comparing this with the conformal transformations of X^μ:

$$\delta_v X^\mu = -\varepsilon v \partial X^\mu - \varepsilon v^* \bar{\partial} X^\mu \qquad (2.4.7)$$

we see that that

$$(\delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1}) X \quad \text{is equivalent to} \quad \delta_v X$$

with $v = -2\eta_1\eta_2$. Of course the quadratic form of ε in our derivation of $(\delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1})$ has to be adjusted accordingly.

For the $\psi, \tilde{\psi}$ fields, we need to first figure out how $\psi, \tilde{\psi}$ transform under conformal trans.

~~a~~. We would like to compute $T\psi$, $T\tilde{\psi}$. First we need an expression for T. In the ψ_1, ψ_2 theory, we have

$$T = -\frac{1}{2}[\psi_1 \partial \psi_1 + \psi_2 \partial \psi_2].$$

Now define $\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$, $\tilde{\psi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2)$.

then $\psi \partial \tilde{\psi} + \tilde{\psi} \partial \psi = \psi_1 \partial \psi_2 + \psi_2 \partial \psi_1$, thus

$$T = -\frac{1}{2}[\psi \partial \tilde{\psi} + \tilde{\psi} \partial \psi]$$

Using $\psi \tilde{\psi} \sim \frac{1}{z}$ (I think this is different from the one Polchinski had on (10.1.7), but this is what was given in section 2.5).

We have $T\psi(0) = -\frac{1}{2}\psi(-\frac{1}{z^2}) = \frac{1}{2z^2}\psi$

$$T\tilde{\psi}(0) = -\frac{1}{2}\tilde{\psi}(-\frac{1}{z^2}) = \frac{1}{2z^2}\tilde{\psi}$$

$$\Rightarrow \psi^{(1)} = \frac{1}{2}\psi \qquad \tilde{\psi}^{(1)} = \frac{1}{2}\tilde{\psi}$$

Again, using Ward identity's constraint on CFTs, (2.4.11, 12) we have

$$\delta_v \psi = -\frac{\varepsilon}{2}\partial v \psi$$

$$\delta_v \tilde{\psi} = -\frac{\varepsilon}{2}\bar{\partial} v \tilde{\psi}.$$

for conformal transformation parameterized by $v(z)$.

Now we go to superconformal transformations,

$$\delta_{\eta_2}\psi = -\varepsilon\sqrt{\tfrac{2}{\alpha'}}\,\eta_2\,\partial X$$

$$\delta_{\eta_1}\delta_{\eta_2}\psi = -\varepsilon\sqrt{\tfrac{2}{\alpha'}}\,\eta_2\,\partial\left(\delta_{\eta_1} X\right)$$

$$= -\varepsilon\sqrt{\tfrac{2}{\alpha'}}\,\eta_2\,\partial\left[\varepsilon\sqrt{\tfrac{\alpha'}{2}}\left(\eta_1\psi + \eta_1^*\tilde{\psi}\right)\right]$$

$$= -\varepsilon^2\,\eta_2\left[(\partial\eta_1)\psi + \eta_1(\partial\psi)\right]$$

$$= -\varepsilon^2\left[\eta_2(\partial\eta_1)\psi + \eta_2\eta_1(\partial\psi)\right]$$

$$\Rightarrow \delta_{\eta_2}\delta_{\eta_1}\psi = -\varepsilon^2\left[\eta_1(\partial\eta_2)\psi + \eta_1\eta_2(\partial\psi)\right]$$

$$(\delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1})\psi = -\varepsilon^2\Big\{\left[\eta_2(\partial\eta_1) - \eta_1(\partial\eta_2)\right]\psi + (\eta_2\eta_1 - \eta_1\eta_2)\,\partial\psi\Big]$$

$$= -\varepsilon^2\Big\{\left[-\eta_1(\partial\eta_2) - (\partial\eta_1)\eta_2\right]\psi - 2\eta_1\eta_2\,\partial\psi\Big\}$$

$$= -\varepsilon^2\left[-\partial[\eta_1\eta_2]\,\psi - 2\eta_1\eta_2\,\partial\psi\right]$$

$$= -\varepsilon^2 \left[-\partial [\eta_1 \eta_2] \psi \right] + O(\partial \psi)$$

Let $v = -2\eta_1 \eta_2$

$$= -\varepsilon^2 \frac{\partial v}{2} \psi + O(\partial \psi)$$

Rescaling $\varepsilon^2 \to \varepsilon$ gives

$$-\frac{\varepsilon}{2} \partial v \psi + \boxed{O(\partial \psi)}$$

this term is not quite the EQM, somewhere went wrong?

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8-29-2024

Polchinski 10.2

(b) Lemma: $\delta_{\eta_1}\delta_{\eta_2} + \delta_{\eta_2}\delta_{\eta_1} = 0$, that is, superconformal transformations anticommute

Proof: By direct computation using what we computed in part (a)

$$(\delta_{\eta_1}\delta_{\eta_2} + \delta_{\eta_2}\delta_{\eta_1})X = -\varepsilon^2\left[(\eta_2\eta_1 + \eta_1\eta_2)\partial X + (\eta_2^*\eta_1^* + \eta_1^*\eta_2^*)\bar\partial X\right]$$

$$= 0.$$

$$(\delta_{\eta_1}\delta_{\eta_2} + \delta_{\eta_2}\delta_{\eta_1})\psi = -\varepsilon^2\left[\left[\eta_2(\partial\eta_1) + \eta_1(\partial\eta_2)\right]\psi + (\eta_2\eta_1 + \eta_1\eta_2)\partial\psi\right]$$

$$= -\varepsilon^2\partial\left[(\eta_2\eta_1 + \eta_1\eta_2)\psi\right]$$

$$= 0. \qquad \square$$

Now consider $[\delta_{\eta_0}, \delta_v]$,

by (10.1.11) or what we showed in part (a),

$$\delta_v = \delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1} \qquad \text{where } -2\eta_1\eta_2 = v,$$

so we can write $[\delta_{\eta_0}, \delta_v]$ with purely superconformal variations

$$[\delta_{\eta_0}, \delta_v] = [\delta_{\eta_0}, \delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1}]$$

$$= \delta_{\eta_0}\delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1}\delta_{\eta_0} - \delta_{\eta_1}\delta_{\eta_2}\delta_{\eta_0} + \delta_{\eta_2}\delta_{\eta_1}\delta_{\eta_0}$$

For notational simplicity we write them as

$$012 - 210 - 120 + 210$$

~~Now add~~ It's probably nicer to work with 3 than 0, so we rename $\delta_{\eta_0} \to \delta_{\eta_3}$, so we have

$$312 - 213 - 123 + 213$$

$$= 312 + 213 - \{2,1\}3$$

recall $\{\delta_{\eta_2}, \delta_{\eta_1}\} = 0$.

$$= 312 + 213$$

So we are now interested in the composition

$$\delta_{\eta_3}\delta_{\eta_1}\delta_{\eta_2} + \delta_{\eta_2}\delta_{\eta_1}\delta_{\eta_3}.$$

We compute this explicitly,

$$\delta_{\eta_1}\delta_{\eta_2} X = -\varepsilon^2 \left[\eta_2\eta_1 \partial X + \eta_2^*\eta_1^* \bar{\partial} X\right]$$

$$\delta_{\eta_3}\delta_{\eta_1}\delta_{\eta_2} X = -\varepsilon^2 \left[\eta_2\eta_1 \partial(\delta_{\eta_3} X) + \eta_2^*\eta_1^* \bar{\partial}(\delta_{\eta_3} X)\right]$$

$$= -\varepsilon^3 \sqrt{\frac{\alpha'}{2}} \Big[\eta_2\eta_1 \partial\left[\eta_3\psi + \eta_3^*\tilde{\psi}\right] + \eta_2^*\eta_1^* \bar{\partial}\left[\eta_3\psi + \eta_3^*\tilde{\psi}\right]\Big]$$

$$= -\varepsilon^3 \sqrt{\frac{\alpha'}{2}} \Big[\eta_2\eta_1(\partial\eta_3)\psi + \eta_2\eta_1\eta_3(\partial\psi) + \eta_2^*\eta_1^*(\bar{\partial}\eta_3^*)\tilde{\psi} + \eta_2^*\eta_1^*\eta_3^*(\bar{\partial}\tilde{\psi})\Big]$$

$$\Rightarrow \left[\delta_{\eta_3}\delta_{\eta_1}\delta_{\eta_2} + \delta_{\eta_2}\delta_{\eta_1}\delta_{\eta_3}\right] X$$

$$= -\varepsilon^3 \sqrt{\frac{\alpha'}{2}} \Big\{ \left[\eta_2\eta_1(\partial\eta_3) + \eta_3\eta_1(\partial\eta_2)\right]\psi + \left[\eta_2\eta_1\eta_3 + \eta_3\eta_1\eta_2\right]\partial\psi + \left[\eta_2^*\eta_1^*(\bar{\partial}\eta_3^*) + \eta_3^*\eta_1^*(\bar{\partial}\eta_2^*)\right]\tilde{\psi} + \left[\eta_2^*\eta_1^*\eta_3^* + \eta_3^*\eta_1^*\eta_2^*\right]\bar{\partial}\tilde{\psi} \Big\}$$

Using anticommutivity: $\eta_2\eta_1\eta_3 = -\eta_2\eta_3\eta_1 = \eta_3\eta_2\eta_1$

$$= -\eta_3\eta_1\eta_2$$

The derivative terms on ψ vanish and we have

$$\left[\delta_{\eta_3}\delta_{\eta_1}\delta_{\eta_2} + \delta_{\eta_2}\delta_{\eta_1}\delta_{\eta_3}\right] X$$

$$= -\varepsilon^3 \sqrt{\frac{\alpha'}{2}} \left\{ \left[\eta_2\eta_1(\partial\eta_3) + \eta_3\eta_1(\partial\eta_2)\right]\psi + \left[\eta_2^*\eta_1^*(\bar\partial\eta_3^*) + \eta_3^*\eta_1^*(\bar\partial\eta_2^*)\right]\tilde\psi \right\}$$

which is a superconformal transformation. (10.1.10 a)

Now we look at $\left[\delta_{\eta_3}\delta_{\eta_1}\delta_{\eta_2} + \delta_{\eta_2}\delta_{\eta_1}\delta_{\eta_3}\right]\psi$

$$\delta_{\eta_1}\delta_{\eta_2}\psi = -\varepsilon^2\left[\eta_2(\partial\eta_1)\psi + \eta_2\eta_1(\partial\psi)\right]$$

$$\delta_{\eta_3}\delta_{\eta_1}\delta_{\eta_2}\psi = -\varepsilon^2\left[\eta_2(\partial\eta_1)(\delta_{\eta_3}\psi) + \eta_2\eta_1\,\partial(\delta_{\eta_3}\psi)\right]$$

$$= \varepsilon^3\sqrt{\frac{2}{\alpha'}}\left\{\eta_2(\partial\eta_1)\,\eta_3(\partial X) + \eta_2\eta_1\,\partial\left[\eta_3\,\partial X\right]\right\}$$

$$= \varepsilon^3\sqrt{\frac{2}{\alpha'}}\left\{\begin{array}{l}\eta_2(\partial\eta_1)\eta_3(\partial X) + \\ \eta_2\eta_1(\partial\eta_3)(\partial X) + \\ \eta_2\eta_1\eta_3(\partial^2 X)\end{array}\right\}$$

$$\Rightarrow \left[\delta_{\eta_3}\delta_{\eta_1}\delta_{\eta_2} + \delta_{\eta_2}\delta_{\eta_1}\delta_{\eta_3}\right]\psi$$

$$= \varepsilon^3\sqrt{\frac{2}{\alpha'}}\left\{\left[\eta_2(\partial\eta_1)\eta_3 + \eta_2\eta_1(\partial\eta_3) + \eta_3(\partial\eta_1)\eta_2 + \eta_3\eta_1(\partial\eta_2)\right] \times \partial X + \left[\eta_2\eta_1\eta_3 + \eta_3\eta_1\eta_2\right]\partial^2 X\right\}$$

$$= \varepsilon^2\sqrt{\frac{2}{\alpha'}}\left[\eta_2\,\partial(\eta_1\eta_3) + \eta_3\,\partial(\eta_1\eta_2)\right]\partial X$$

which is of the form of superconformal transformation (10.1.10b).

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