Polchinski 10.1(a)

$$T_F = i\sqrt{\tfrac{2}{\alpha'}}\,\psi^\mu(z)\,\partial X_\mu(z) \qquad \tilde{T}_F = i\sqrt{\tfrac{2}{\alpha'}}\,\tilde{\psi}^\mu(\bar{z})\,\bar{\partial} X_\mu(\bar{z})$$

$$T_F X^\nu(0) = i\sqrt{\tfrac{2}{\alpha'}}\,\psi^\mu(z)\,\partial X_\mu(z)\,X^\nu(0) \qquad \tilde{T}_F X^\nu(0) = i\sqrt{\tfrac{2}{\alpha'}}\,\tilde{\psi}^\mu(\bar{z})\,\bar{\partial} X_\mu(\bar{z})\,X^\nu(0)$$

$$= i\sqrt{\tfrac{2}{\alpha'}}\,\psi^\mu(z)\,\partial\left[-\tfrac{\alpha'}{2}\ln|z|^2\right]\eta_\mu^{\ \nu} \qquad = i\sqrt{\tfrac{2}{\alpha'}}\,\tilde{\psi}^\mu(\bar{z})\,\bar{\partial}\left[-\tfrac{\alpha'}{2}\ln|z|^2\right]\eta_\mu^{\ \nu}$$

$$= i\sqrt{\tfrac{2}{\alpha'}}\,\psi^\mu(z)\left(\tfrac{-\alpha'}{2}\right)\tfrac{1}{z}\,\eta_\mu^{\ \nu} \qquad = i\sqrt{\tfrac{2}{\alpha'}}\,\tilde{\psi}^\mu(\bar{z})\left(\tfrac{-\alpha'}{2}\right)\tfrac{1}{\bar{z}}\,\eta_\mu^{\ \nu}$$

$$= \boxed{-i\sqrt{\tfrac{\alpha'}{2}}\,\psi^\nu(z)\,\tfrac{1}{z}} \qquad = \boxed{-i\sqrt{\tfrac{\alpha'}{2}}\,\tilde{\psi}^\nu(\bar{z})\,\tfrac{1}{\bar{z}}}$$

$$T_F\,\psi^\nu(0) = i\sqrt{\tfrac{2}{\alpha'}}\,\partial X_\mu(z)\,\psi^\mu(z)\,\psi^\nu(0) \qquad \tilde{T}_F\,\tilde{\psi}^\nu(0) = i\sqrt{\tfrac{2}{\alpha'}}\,\bar{\partial} X_\mu(\bar{z})\,\tilde{\psi}^\mu(\bar{z})\,\tilde{\psi}^\nu(0)$$

$$= i\sqrt{\tfrac{2}{\alpha'}}\,\partial X_\mu(z)\left[\eta^{\mu\nu}\tfrac{1}{z}\right] \qquad = i\sqrt{\tfrac{2}{\alpha'}}\,\bar{\partial} X_\mu(\bar{z})\left[\eta^{\mu\nu}\tfrac{1}{\bar{z}}\right]$$

$$= \boxed{i\sqrt{\tfrac{2}{\alpha'}}\,\partial X^\nu(z)\,\tfrac{1}{z}} \qquad = \boxed{i\sqrt{\tfrac{2}{\alpha'}}\,\bar{\partial} X^\nu(\bar{z})\,\tfrac{1}{\bar{z}}}$$

(b) One recalls (2.4.11), (2.4.12), the variation of $\mathcal{A}$ can be given by terms in expansion $T\mathcal{A}$:

$$\delta \mathcal{A}(z,\bar{z}) = -\varepsilon \sum_{n=0}^{\infty} \frac{1}{n!}\left[\partial^n v\, \mathcal{A}^{(n)}(z,\bar{z}) + \bar{\partial}^n v^* \tilde{\mathcal{A}}^{(n)}(z,\bar{z})\right] \tag{2.4.12}$$

has $\mathcal{A}^{(n)}$ determined by

$$T(z)\mathcal{A}(0,0) \sim \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \mathcal{A}^{(n)}(0,0). \tag{2.4.11}$$

Looking at the form of the $T_F X$, $\tilde{T}_F X$, $T_F \psi$, $\tilde{T}_F \tilde{\psi}$ that were determined before, we see that the only relevant terms here are the $n=1$ terms; specifically

$$X^{\mu(1)} = -i\sqrt{\frac{\alpha'}{2}}\,\psi^\mu(z) \qquad \tilde{X}^{\mu(1)} = -i\sqrt{\frac{\alpha'}{2}}\,\tilde{\psi}^\mu(\bar{z})$$

$$\psi^{\mu(1)} = i\sqrt{\frac{2}{\alpha'}}\,\partial X^\mu(z) \qquad \tilde{\psi}^{\mu(1)} = i\sqrt{\frac{2}{\alpha'}}\,\bar{\partial} X^\mu(\bar{z})$$

$$\Rightarrow \delta X^\mu = -\varepsilon\left[v\left(-i\sqrt{\frac{\alpha'}{2}}\,\psi^\mu(z)\right) + v^*\left(-i\sqrt{\frac{\alpha'}{2}}\,\tilde{\psi}^\mu(\bar{z})\right)\right]$$

$$\varepsilon^{-1}\sqrt{\frac{2}{\alpha'}}\,\delta X^\mu = iv\,\psi^\mu(z) + iv^*\,\tilde{\psi}^\mu(\bar{z})$$

Replacing v with $\frac{\eta(z)}{i}$, v^* with $\frac{\eta^*(z)}{i}$ yields

$$\varepsilon^{-1}\sqrt{\frac{2}{\alpha'}}\,\delta X^\mu = \eta(z)\psi^\mu(z) + \eta^*(z)\tilde{\psi}^\mu(\bar{z}) \tag{10.10.10a}$$

The same procedure carries for ψ:

$$\delta\psi^\mu = -\epsilon\, v\, i\sqrt{\frac{2}{\alpha'}}\,\partial X^\mu(z)$$

$$\Rightarrow \quad \epsilon^{-1}\sqrt{\frac{\alpha'}{2}}\,\delta\psi^\mu = -\eta(z)\,\partial X^\mu(z) \qquad (10.1.10b)$$

$$\delta\tilde{\psi}^\mu = -\epsilon\, v^*\, i\sqrt{\frac{2}{\alpha'}}\,\bar\partial X^\mu(\bar z)$$

$$\Rightarrow \quad \epsilon^{-1}\sqrt{\frac{\alpha'}{2}}\,\delta\tilde\psi^\mu = -\eta^*(z)\,\bar\partial X^\mu(\bar z) \qquad (10.1.10c)$$

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