Schwartz 15.2

Reference: Schwartz 12.61 → 12.68

Take fermionic Lagr: $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$

$$T_{\mu\nu} = i\bar{\psi}\gamma_\mu\partial_\nu\psi - g_{\mu\nu}[\bar{\psi}(i\not{\partial} - m)\psi]$$

$$\mathcal{E} = T_{00} = \bar{\psi}(i\gamma^i\partial_i + m)\psi$$

Apply
$$m\psi = -\not{p}\psi$$
$$= i\gamma^\mu\partial_\mu\psi$$
$$= i[\gamma^0\partial_0 - \vec{\gamma}\cdot\vec{\nabla}]\psi$$

$$\Rightarrow \quad \mathcal{E} = \bar{\psi}(i\gamma^0\partial_0\psi)$$

Fermionic fields are

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p^s u_p^s e^{-ipx} + b_p^{s\dagger} v_p^s e^{ipx})$$

$$\bar{\psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p^{s\dagger}\bar{u}_p^s e^{ipx} + b_p^s \bar{v}_p^s e^{-ipx})$$

$$i\partial_t\psi = i\sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}[-i\omega_p a_p^s u_p^s e^{-ipx} + i\omega_p b_p^{s\dagger} v_p^s e^{ipx}]$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}[a_p^s u_p^s e^{-ipx} - b_p^{s\dagger} v_p^s e^{ipx}]$$

$$\mathcal{E} = \int \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{p_1}}} \sqrt{\frac{\omega_{p_2}}{2}}$$

$$\times \sum_{s_1, s_2} \left[a_{p_1}^{s_1 \dagger} \bar{u}_{p_1}^{s_1} e^{ip_1 x} + i\omega_{p_1} b_{p_1}^{s_1} \bar{v}_{p_1}^{s_1} e^{ip_1 x} \right]$$

$$\times \gamma^0 \left[a_{p_2}^{s_2} u_{p_2}^{s_2} e^{-ip_2 x} - b_{p_2}^{s_2 \dagger} v_{p_2}^{s_2} e^{ip_2 x} \right]$$

$$\bar{u} = u^\dagger \gamma^0 \quad \Rightarrow \quad \bar{u}_s \gamma^0 u_{s'} = u_s^\dagger \gamma^0 \gamma^0 u_{s'} = u_s^\dagger u_{s'}$$

$$= 2\omega_p \delta_{ss'}$$

$$= \sum_s \int \frac{d^3 p}{(2\pi)^3} \omega_p \left[a_p^{s\dagger} a_p^s - b_p^s b_p^{s\dagger} \right]$$

$b_p, b_p^\dagger$ anticommutes:

$$b_p b_{p'}^\dagger + b_p^\dagger b_{p'} = (2\pi)^3 \delta^3(p - p')$$

$$b_p^s b_p^{s\dagger} = (2\pi)^3 \delta^3(p - p') - b_p^\dagger b_p$$

$$\mathcal{E} = \sum_s \int \frac{d^3 p}{(2\pi)^3} \omega_p \left[a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p - (2\pi)^3 \delta^3(0) \right]$$

$$= \sum_s \left[\int \frac{d^3 p}{(2\pi)^3} \omega_p \left[a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p \right] - 1 \right]$$

We might as well interpret $\mathcal{E} = \mathcal{E}_a + \mathcal{E}_b$

$$\mathcal{E}_a = \int \frac{d^3 p}{(2\pi)^3} \omega_p a_p^{s\dagger} a_p^s - \frac{1}{2} \qquad \mathcal{E}_b = \int \frac{d^3 p}{(2\pi)^3} \omega_p b_p^{s\dagger} b_p^s - \frac{1}{2}$$

summation on s implied.

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