Schwartz

10.4 (a) Majorana Rep:

$$\gamma_0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} i\sigma_3 & \\ & i\sigma_3 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} -i\sigma_1 & \\ & -i\sigma_1 \end{pmatrix}.$$

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$S^{01} = \frac{i}{4}[\gamma_0, \gamma_1] = \frac{i}{4}\left[\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\begin{pmatrix} i\sigma_3 & \\ & i\sigma_3 \end{pmatrix} - \begin{pmatrix} i\sigma_3 & \\ & i\sigma_3 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\right]$$

$$= \frac{i \cdot i}{4}\left[\begin{pmatrix} & \sigma_2\sigma_3 \\ \sigma_2\sigma_3 & \end{pmatrix} - \begin{pmatrix} & \sigma_3\sigma_2 \\ \sigma_3\sigma_2 & \end{pmatrix}\right]$$

$$= \frac{-1}{4}\begin{bmatrix} & [\sigma_2, \sigma_3] \\ [\sigma_2, \sigma_3] & \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} & \sigma_1 \\ \sigma_1 & \end{bmatrix} 2i$$

$$= \frac{-i}{2}\begin{bmatrix} & \sigma_1 \\ \sigma_1 & \end{bmatrix}$$

$$S^{02} = \frac{i}{4}[\gamma_0, \gamma_2] = \frac{i}{4}\left[\begin{pmatrix} & \sigma_2 \\ \sigma_2 & \end{pmatrix}\begin{pmatrix} & -\sigma_2 \\ \sigma_2 & \end{pmatrix} - \begin{pmatrix} & -\sigma_2 \\ \sigma_2 & \end{pmatrix}\begin{pmatrix} & \sigma_2 \\ \sigma_2 & \end{pmatrix}\right]$$

$$= \frac{i}{4}\left[\begin{pmatrix} \sigma_2^2 & \\ & -\sigma_2^2 \end{pmatrix} - \begin{pmatrix} -\sigma_2^2 & \\ & \sigma_2^2 \end{pmatrix}\right]$$

$$= \frac{i}{2}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$$

$$S^{03} = \frac{i}{4}[\gamma_0, \gamma_3] = \frac{i}{4}\left[\begin{pmatrix} & \sigma_2 \\ \sigma_2 & \end{pmatrix}\begin{pmatrix} -i\sigma_1 & \\ & -i\sigma_1 \end{pmatrix} - \begin{pmatrix} -i\sigma_1 & \\ & -i\sigma_1 \end{pmatrix}\begin{pmatrix} & \sigma_2 \\ \sigma_2 & \end{pmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix} & \sigma_2 \\ \sigma_2 & \end{pmatrix}\begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix} - \begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix}\begin{pmatrix} & \sigma_2 \\ \sigma_2 & \end{pmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix} & \sigma_2\sigma_1 \\ \sigma_2\sigma_1 & \end{pmatrix} - \begin{pmatrix} & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \end{pmatrix}\right]$$

$$= \frac{i}{2}\begin{bmatrix} & \sigma_3 \\ \sigma_3 & \end{bmatrix}$$

$$S^{12} = \frac{i}{4}[\gamma_1, \gamma_2] = \frac{-1}{4}\left[\begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix}\begin{pmatrix} & -\sigma_2 \\ \sigma_2 & \end{pmatrix} - \begin{pmatrix} & -\sigma_2 \\ \sigma_2 & \end{pmatrix}\begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix}\right]$$

$$= \frac{-1}{4}\left[\begin{pmatrix} & -\sigma_3\sigma_2 \\ \sigma_3\sigma_2 & \end{pmatrix} - \begin{pmatrix} & -\sigma_2\sigma_3 \\ \sigma_2\sigma_3 & \end{pmatrix}\right]$$

$$= \frac{-1}{4}\begin{bmatrix} & \sigma_2\sigma_3 - \sigma_3\sigma_2 \\ \sigma_3\sigma_2 - \sigma_2\sigma_3 & \end{bmatrix}$$

$$= \frac{-1}{4}\begin{bmatrix} & 2i\sigma_1 \\ -2i\sigma_1 & \end{bmatrix}$$

$$= \frac{i}{2}\begin{bmatrix} & -\sigma_1 \\ \sigma_1 & \end{bmatrix}$$

$$S^{13} = \frac{i}{4}[\gamma_1, \gamma_3] = \frac{i}{4}\left[\begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix}\begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix} - \begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix}\begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix}\right]$$

$$= \frac{i}{4}\begin{bmatrix} \sigma_3\sigma_1 - \sigma_1\sigma_3 & \\ & \sigma_3\sigma_1 - \sigma_1\sigma_3 \end{bmatrix}$$

$$= \frac{i}{4}\begin{bmatrix} 2i\sigma_2 & \\ & 2i\sigma_2 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} \sigma_2 & \\ & \sigma_2 \end{bmatrix}$$

$$S^{23} = \frac{i}{4}[\gamma_2, \gamma_3] = \frac{1}{4}\left[\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix} - \begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix}\begin{pmatrix} & -\sigma_2 \\ \sigma_2 & \end{pmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix} & -\sigma_2\sigma_1 \\ \sigma_2\sigma_1 & \end{pmatrix} - \begin{pmatrix} & -\sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \end{pmatrix}\right]$$

$$= \frac{1}{4}\begin{bmatrix} & \sigma_1\sigma_2 - \sigma_2\sigma_1 \\ \sigma_2\sigma_1 - \sigma_1\sigma_2 & \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} & 2i\sigma_3 \\ -2i\sigma_3 & \end{bmatrix}$$

$$= \frac{i}{2}\begin{bmatrix} & \sigma_3 \\ -\sigma_3 & \end{bmatrix}$$

To summarize, we found the independent components of $S_{\mu\nu}$ as

$$S^{01} = \frac{1}{2}\begin{bmatrix} & -i\sigma_1 \\ -i\sigma_1 & \end{bmatrix}$$

$$S^{02} = \frac{1}{2}\begin{bmatrix} i & \\ & -i \end{bmatrix}$$

$$S^{03} = \frac{1}{2}\begin{bmatrix} & i\sigma_3 \\ i\sigma_3 & \end{bmatrix}$$

$$S^{12} = \frac{1}{2}\begin{bmatrix} & -i\sigma_1 \\ i\sigma_1 & \end{bmatrix}$$

$$S^{13} = \frac{1}{2}\begin{bmatrix} -\sigma_2 & \\ & -\sigma_2 \end{bmatrix}$$

$$S^{23} = \frac{1}{2}\begin{bmatrix} & i\sigma_3 \\ -i\sigma_3 & \end{bmatrix}$$

(b)

$$\vec{J}^2 = \vec{J}_1^{\,2} + \vec{J}_2^{\,2} + \vec{J}_3^{\,2}$$

$$(S_{12})^2 = \frac{1}{4}\begin{bmatrix} & -i\sigma_1 \\ i\sigma_1 & \end{bmatrix}\begin{bmatrix} & -i\sigma_1 \\ i\sigma_1 & \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} \sigma_1^2 & \\ & \sigma_1^2 \end{bmatrix}$$

$$= \frac{1}{4}\mathbb{1}$$

$$(S_{13})^2 = \frac{1}{4}\begin{bmatrix} -\sigma_2 & \\ & -\sigma_2 \end{bmatrix}\begin{bmatrix} -\sigma_2 & \\ & -\sigma_2 \end{bmatrix}$$

$$= \frac{1}{4}\mathbb{1}$$

$$(S_{23})^2 = \frac{1}{4}\begin{bmatrix} & i\sigma_3 \\ -i\sigma_3 & \end{bmatrix}\begin{bmatrix} & i\sigma_3 \\ -i\sigma_3 & \end{bmatrix}$$

$$= \frac{1}{4}\mathbb{1}.$$

$\Rightarrow \boxed{\vec{J}^2 = \frac{3}{4}\mathbb{1}.}$ for Majorana Rep.

The left-handed Weyl Spinor has

$$S^{12} = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad S^{13} = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S^{23} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\vec{J}^2 = (S_{12})^2 + (S_{13})^2 + (S_{23})^2$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \boxed{\frac{3}{4}\mathbb{1}}$$

$\vec{J}^2$ is the same for Majorana and left-handed Weyl.

The interpretation is that it's equal to the intrinsic total spin in non-relativistic QM, because

$$\frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4} = s(s+1).$$

(c) $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$

In Majoranna Rep. it's

$$i\begin{pmatrix} & \sigma_2 \\ \sigma_2 & \end{pmatrix}\begin{pmatrix} i\sigma_3 & \\ & i\sigma_3 \end{pmatrix}\begin{pmatrix} & -\sigma_2 \\ \sigma_2 & \end{pmatrix}\begin{pmatrix} -i\sigma_1 & \\ & -i\sigma_1 \end{pmatrix}$$

$$= \begin{pmatrix} & i\sigma_2\sigma_3 \\ i\sigma_2\sigma_3 & \end{pmatrix}\begin{pmatrix} & -\sigma_2\sigma_1 \\ \sigma_2\sigma_1 & \end{pmatrix}$$

$$= i\begin{pmatrix} \sigma_2\sigma_3\sigma_2\sigma_1 & \\ & -\sigma_2\sigma_3\sigma_2\sigma_1 \end{pmatrix}$$

$$= i\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

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3.15.2024