Griffiths

4.23. (a) $S_z = \hbar \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}.$

(b) $S_+ \propto \begin{pmatrix} 0 & a & \\ & 0 & b \\ & & 0 \end{pmatrix}, \quad S_- \propto \begin{pmatrix} 0 & & \\ a & 0 & \\ & b & 0 \end{pmatrix}.$

$S_\pm |s\,m\rangle = \hbar \sqrt{s(s+1) - m(m\pm 1)}\; |s\,(m\pm 1)\rangle.$

$\Rightarrow S_+ |1\,-1\rangle = \hbar \sqrt{2 - (-1)(0)}\; |1\,0\rangle$

$= \hbar\sqrt{2}\, |1\,0\rangle.$

$S_+ |1\,0\rangle = \hbar\sqrt{2}\, |1\,1\rangle.$

$S_- |1\,1\rangle = \hbar\sqrt{2}\, |1\,0\rangle$

$S_- |1\,0\rangle = \hbar\sqrt{2}\, |1\,-1\rangle.$

$\Rightarrow S_+ = \hbar\sqrt{2} \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \quad S_- = \hbar\sqrt{2} \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & 1 & 0 \end{pmatrix}.$

(c) $S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & \\ 1 & 0 & 1 \\ & 1 & 0 \end{pmatrix}.$

$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{i\sqrt{2}} \begin{pmatrix} 0 & 1 & \\ -1 & 0 & 1 \\ & -1 & 0 \end{pmatrix}.$

(d) letting $|\frac{3}{2}\,\frac{3}{2}\rangle$, $|\frac{3}{2}\,\frac{1}{2}\rangle$, $|\frac{3}{2}\,-\frac{1}{2}\rangle$, $|\frac{3}{2}\,-\frac{3}{2}\rangle$, have orthonormal basis $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$.

then $S_z = \frac{\hbar}{2}\begin{pmatrix}3 & & & \\ & 1 & & \\ & & -1 & \\ & & & 3\end{pmatrix}$.

$$S_+|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle = \hbar\sqrt{\tfrac{3}{2}(\tfrac{5}{2}) - (-\tfrac{3}{2})(-\tfrac{1}{2})}\;|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle$$

$$= \frac{\hbar}{\cancel{2}}\sqrt{\frac{15}{4} \cancel{+} - \frac{3}{4}}\;|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle$$

$$= \frac{\hbar}{\cancel{2}}\sqrt{\frac{\cancel{18}\,12}{4}}\;|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle.$$

$$S_+|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{3}{2}(\tfrac{5}{2}) - (-\tfrac{1}{2})(\tfrac{1}{2})}.\;|\tfrac{3}{2}\,\tfrac{1}{2}\rangle$$

$$= \hbar\cancel{\sqrt{\frac{16}{4}}}\sqrt{(4)}\;|\tfrac{3}{2}\,\tfrac{1}{2}\rangle.$$

$$S_+|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{3}{2}\,\tfrac{5}{2} - (\tfrac{1}{2})(\tfrac{3}{2})}\;|\tfrac{3}{2}\,\tfrac{3}{2}\rangle$$

$$= \hbar\sqrt{\frac{12}{4}}\;|\tfrac{3}{2}\,\tfrac{3}{2}\rangle.$$

$$S_-\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle = \hbar\sqrt{\tfrac{15}{4} - \left(\tfrac{3}{2}\right)\left(\tfrac{1}{2}\right)}\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle$$

$$= \hbar\sqrt{\tfrac{12}{4}}\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle.$$

$$S_-\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle = \hbar\sqrt{\tfrac{15}{4} - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)}\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle$$

$$= \hbar\sqrt{4}\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle$$

$$S_-\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle = \hbar\sqrt{\tfrac{15}{4} - \left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)}\left|\tfrac{3}{2}\,-\tfrac{3}{2}\right\rangle$$

$$= \hbar\sqrt{\tfrac{12}{4}}\left|\tfrac{3}{2}\,-\tfrac{3}{2}\right\rangle.$$

$$\Rightarrow S_+ = \hbar\begin{pmatrix} 0 & \sqrt{3} & & \\ & 0 & 2 & \\ & & 0 & \sqrt{3} \\ & & & 0 \end{pmatrix}.$$

$$S_- = \hbar\begin{pmatrix} 0 & & & \\ \sqrt{3} & 0 & & \\ & 2 & 0 & \\ & & \sqrt{3} & 0 \end{pmatrix}$$

$$S_x = \tfrac{1}{2}(S_+ + S_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & & \\ \sqrt{3} & 0 & 2 & \\ & 2 & 0 & \sqrt{3} \\ & & \sqrt{3} & 0 \end{pmatrix}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{3} & & \\ -\sqrt{3} & 0 & 2 & \\ & -2 & 0 & \sqrt{3} \\ & & -\sqrt{3} & 0 \end{pmatrix}$$