Schutz 5.7

$$\Lambda^{\alpha'}{}_{\beta} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}} \qquad x^{\alpha'} = \{r, \theta\}$$
$$x^{\beta} = \{x, y\}.$$

$$r = x^2 + y^2, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

$$\frac{\partial r}{\partial x} = 2x \qquad \frac{\partial \tan^{-1}\left(\frac{y}{x}\right)}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial r}{\partial y} = 2y$$

$$\frac{\partial \tan^{-1}\left(\frac{y}{x}\right)}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right).$$

$$\frac{\partial}{\partial x}\tan^{-1}\left(\frac{y}{x}\right) = \frac{-1}{1+\frac{y^2}{x^2}}\frac{y}{x^2} = -\frac{x^2}{x^2+y^2}\frac{y}{x^2}$$

$$= \boxed{-\frac{y}{x^2+y^2}}$$

$$\frac{\partial}{\partial y}\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x} = \frac{x^2}{x^2+y^2}\frac{1}{x}$$

$$= \boxed{\frac{x}{x^2+y^2}}$$

$$\boxed{\begin{array}{ll} \Lambda^{r}{}_{x} = 2x & \Lambda^{r}{}_{y} = 2y \\ \Lambda^{\theta}{}_{x} = -\frac{y}{x^2+y^2} & \Lambda^{\theta}{}_{y} = \frac{x}{x^2+y^2} \end{array}}$$

$x = r\cos\theta$, $y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

$\Rightarrow$

$$\Lambda^x_r = \cos\theta \qquad \Lambda^x_\theta = -r\sin\theta$$

$$\Lambda^y_r = \sin\theta \qquad \Lambda^y_\theta = r\cos\theta$$

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