Schutz 5.12

(a) $\tilde{p}$. $\quad p_\alpha = (x^2+3y,\ y^2+3x)$ Cartesian.

$$p_{\alpha,\beta} = \cancel{p_{x,x}}$$

$$p_{1,1} = 2x$$
$$p_{1,2} = 3$$
$$p_{2,1} = 3$$
$$p_{2,2} = 2y$$

(b) $\Lambda^\alpha{}_{\mu'} = \dfrac{\partial x^\alpha}{\partial x^{\mu'}}$ $\qquad x = r\cos\theta,\ y = r\sin\theta$

$$\Lambda^1{}_{1'} = \frac{\partial x}{\partial r} = \cos\theta, \quad \Lambda^1{}_{2'} = \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\Lambda^2{}_{1'} = \frac{\partial y}{\partial r} = \sin\theta, \quad \Lambda^2{}_{2'} = \frac{\partial y}{\partial \theta} = r\cos\theta$$

$\Lambda^\beta{}_{\nu'} \, p_{\alpha,\beta}$ contracts β $= \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix}$

$$\Lambda^\alpha{}_{\mu'} \Lambda^\beta{}_{\nu'} p_{\alpha,\beta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Let A denote $\cos\theta$, B denote $\sin\theta$.

$$\begin{pmatrix} A & B \\ -rB & rA \end{pmatrix} \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} \begin{pmatrix} A & -rB \\ B & rA \end{pmatrix}$$

$$= \begin{pmatrix} 2xA + 3B & 3A + 2yB \\ -2xrB + 3rA & -3rB + 2yrA \end{pmatrix} \begin{pmatrix} A & -rB \\ B & rA \end{pmatrix}$$

$$= \begin{pmatrix} 2xA^2 + 3BA + 3AB + 2yB^2 & -2xrAB - 3rB^2 + 3rA^2 + 2yrAB \\ -2xrAB + 3rA^2 - 3rB^2 + 2yrAB & 2xr^2B^2 - 3r^2AB - 3r^2AB + 2yr^2A^2 \end{pmatrix}$$

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written in form

$$\begin{pmatrix} P_{r;r} & P_{r;\theta} \\ P_{\theta;r} & P_{\theta;\theta} \end{pmatrix}$$

Schutz 5.12

(6) $$\tilde{p} = p_{\mu'}\tilde{\omega}^{\mu'} = p_\alpha \tilde{\omega}^\alpha$$

$$\tilde{\omega}^\alpha = \Lambda^\alpha{}_{\mu'}\tilde{\omega}^{\mu'}$$

$$\Rightarrow \quad p_{\mu'}\tilde{\omega}^{\mu'} = p_\alpha \Lambda^\alpha{}_{\mu'}\tilde{\omega}^{\mu'},$$

$$p_{\mu'} = p_\alpha \Lambda^\alpha{}_{\mu'}$$

$$p_\alpha = (x^2+3y,\ y^2+3x)$$

$$\Lambda^\alpha{}_{\mu'} = \begin{pmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{pmatrix}.$$

$$p_{\mu'} = p_\alpha \Lambda^\alpha{}_{\mu'} = (x^2+3y \quad y^2+3x)\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}$$

$$= (x^2+3y \quad y^2+3x)\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

$$= (x^2+3y \quad y^2+3x)\begin{pmatrix} A & -rB \\ B & rA \end{pmatrix}$$

$$= (Ax^2+3yA+By^2+3xB \quad -rBx^2-rB3x+y^2rA+3xrA)$$

$$= (rA^3+3rAB+rB^3+3rAB,\ -r^3A^2B-3r^2AB+r^3AB^2+3r^2A^2)$$

We check $P_{1';1'}$ component with part (b)

Nonzero components of Γ: $\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}$.

$\Gamma^{r}_{\theta\theta} = -r$.

$P_{\mu',\nu'} = P_{1',1'} = A^3 + 6AB + B^3$

$P_{1';1'} = A^3 + 6AB + B^3 - \Gamma^{\mu'}_{1'1'} P_{\mu'}$

$= A^3 + 6AB + B^3$

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checks.

Davidson Cheng

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