Goldstein 8.12

$R \equiv$ center of mass $= \dfrac{m_1 q_1 + m_2 q_2}{m_1 + m_2}$

$r \equiv$ separation $= q_1 - q_2$

In $R, r$, the kinetic and potential are

$$T = \frac{m_1 + m_2}{2} \dot{R}^2 + \frac{m_1 m_2}{m_1 + m_2} \dot{r}^2 , \quad V = V(r)$$

$$\Rightarrow \quad L = T - V$$

$$P_R = \frac{\partial L}{\partial \dot{R}} = (m_1 + m_2) R , \quad P_r = \frac{\partial L}{\partial \dot{r}} = \frac{m_1 m_2}{m_1 + m_2} \dot{r}$$

Using $H = T + V$, we have

$$H = \frac{P_R^2}{2(m_1 + m_2)} + \frac{P_r^2 (m_1 + m_2)}{2 (m_1 m_2)} + V(r)$$

$R$ is the cyclic variable, $\Rightarrow \quad \dot{P}_R = 0$.

Removing $R$ by adjusting the zero energy according, we are left with

$$H' = \frac{P_r^2 (m_1 + m_2)}{2 (m_1 m_2)} + V(r)$$

Using the Hami eqm:

$$\dot{P_r} = -\frac{\partial H'}{\partial r} = -\frac{\partial V(r)}{\partial r}$$

Combined with $P_r = \frac{m_1 m_2}{m_1 + m_2}\dot{r}$, we have

the equation:

$$\boxed{\frac{m_1 m_2}{m_1 + m_2}\ddot{r} = -\frac{\partial V(r)}{\partial r}}$$

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