Goldstein 6-3(a)

$$L = \frac{m}{2}(R\sin\theta)^2\dot{\phi}^2 + \frac{m}{2}R^2\dot{\theta}^2 - gmR\cos\theta$$

$$\frac{d}{dt}\frac{dL}{d\dot{\theta}} = mR^2\ddot{\theta} \qquad \frac{dL}{d\theta} = mR^2\sin\theta\cos\theta\,\omega^2 + mgR\sin\theta$$

$$\Rightarrow \text{EQM:} \quad \boxed{mR^2\ddot{\theta} - mR^2\sin\theta\cos\theta\,\omega^2 - mgR\sin\theta = 0.}$$

Physical interpretation: $mR^2\ddot{\theta}$ is torque
$mgR\sin\theta$ is gravitational
$mR^2\sin\theta\cos\theta\,\omega^2$ is fictitional.

Goldstein 6.3(b)

With $mR^2\ddot{\theta} - mR^2 \sin\theta\cos\theta\,\omega^2 - mgR\sin\theta = 0.$

we have $\ddot{\theta} = \sin\theta\cos\theta\,\omega^2 + \frac{g}{R}\sin\theta$

Expand $\ddot{\theta}[\theta]$ around $\theta = \pi$

$\ddot{\theta}[\theta = \pi] = 0$

$$\frac{d\ddot{\theta}}{d\theta}\bigg|_{\theta=\pi} = \left[\cos^2\theta\,\omega^2 - \sin^2\theta\,\omega^2 + \frac{g}{R}\cos\theta\right]\bigg|_{\theta=\pi}$$

$$= \omega^2 - \frac{g}{R}.$$

$\Rightarrow \ddot{\theta} \cong \left(\omega^2 - \frac{g}{R}\right)(\theta - \pi)$ for $\theta - \pi$ small.

For $\omega \geq \sqrt{\frac{g}{R}}$, the direction of $\ddot{\theta}$ is not toward the equilibrium position $\theta = \pi$, which means the motion is not bound. Thus

$$\boxed{\Omega = \sqrt{\frac{g}{R}}}$$

Goldstein 6.3(c)

For $w > \Omega = \sqrt{\frac{g}{R}}$,

$$\ddot{\theta}[\theta] = \sin\theta\cos\theta\, w^2 + \frac{g}{R}\sin\theta$$

Solving for $\ddot{\theta} = 0$ gives $\sin\theta\cos\theta\, w^2 + \frac{g}{R}\sin\theta = 0$,

$$\cos\theta\, w^2 = -\frac{g}{R},$$

$$\cos\theta = -\frac{g}{Rw^2},$$

$$\theta_0 = \cos^{-1}\left[-\frac{g}{Rw^2}\right].$$

Now we check if this equilibrium is stable, we expand $\ddot{\theta}$ in the neighborhood of θ_0.

$$\ddot{\theta}[\theta_0] = 0$$

$$\left.\frac{d\ddot{\theta}}{d\theta}\right|_{\theta_0} = \left.\left[\cos^2\theta\, w^2 - \sin^2\theta\, w^2 + \frac{g}{R}\cos\theta\right]\right|_{\theta_0}$$

$\theta_0 = \cos^{-1}\left[-\frac{g}{Rw^2}\right]$ tells us

$$\left.\cos^2\theta\right|_{\theta_0} = \frac{g^2}{R^2w^4}, \quad \left.\sin^2\theta\right|_{\theta_0} = \left.1-\cos^2\theta\right|_{\theta_0} = 1 - \frac{g^2}{R^2w^4}$$

$$\Rightarrow \left.\frac{d\ddot{\theta}}{d\theta}\right|_{\theta_0} = \frac{g^2}{R^2w^2} + \frac{g^2}{R^2w^2} - w^2 - \frac{g}{R}\left[\frac{g}{Rw^2}\right]$$

$$= \frac{g^2}{R^2w^2} - w^2$$

$$\frac{g^2}{R^2\omega^2} - \omega^2 = \frac{g^2 - R^2\omega^4}{R^2\omega^2}$$

we know $\omega > \sqrt{\frac{g}{R}}$, so $R^2\omega^4 > g^2$, and $\left(\frac{g^2 - R^2\omega^4}{R^2\omega^2}\right) < 0$.

~~Th.~~ Plugging this back,

$$\ddot{\theta} \simeq \left(\frac{g^2 - R^2\omega^4}{R^2\omega^2}\right)(\theta - \theta_0)$$

for $\theta - \theta_0$ small.

$\frac{g^2 - R^2\omega^4}{R^2\omega^2} < 0$ tells us the force is restorative, the motion is bound thus the equilibrium $\theta_0 = \cos^{-1}\left[-\frac{g}{R\omega^2}\right]$ is stable!

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