Goldstein 6.2

$$V = \frac{k}{2}\left[x_2 - (x_1+b)\right]^2 + \frac{k}{2}\left[(x_3-b) - x_2\right]^2 + \frac{k}{2}x_2^2$$

considering small displacements $\eta_2 = x_2$, $\eta_1 = x_1 + b$, $\eta_3 = x_3 - b$.

$$V = \frac{k}{2}(\eta_2 - \eta_1)^2 + \frac{k}{2}(\eta_3 - \eta_2)^2 + \frac{k}{2}\eta_2^2$$

$$= \frac{k}{2}\left[\eta_1^2 + 3\eta_2^2 + \eta_3^2 - 2\eta_1\eta_2 - 2\eta_2\eta_3\right]$$

$$T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_3^2) + \frac{M}{2}\dot{x}_2^2$$

$$= \frac{m}{2}(\dot{\eta}_1^2 + \dot{\eta}_3^2) + \frac{M}{2}\dot{\eta}_2^2$$

$$\overleftrightarrow{V} = \begin{bmatrix} k & -k & \\ -k & 3k & -k \\ & -k & k \end{bmatrix}$$

$$\overleftrightarrow{T} = \begin{bmatrix} m & & \\ & M & \\ & & m \end{bmatrix}$$

$$V - \omega^2 T = \begin{bmatrix} k-\omega^2 m & -k & \\ -k & 3k-\omega^2 M & -k \\ & -k & k-\omega^2 m \end{bmatrix}$$

$$|V - \omega^2 T| = (k-\omega^2 m)\left[(3k-\omega^2 M)(k-\omega^2 m) \cancel{*} - k^2\right]$$

$$+ (k)(-k)(k-\omega^2 m)$$

$$= (k-\omega^2 m)\left[(3k-\omega^2 M)(k-\omega^2 m)-2k^2\right]$$

This gives first solution of ω: $\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$

For the other 2 solutions, just solve for ω^2 in $\left[(3k-\omega^2 M)\cdots 2k^2\right]$

$$(3k-\omega^2 M)(k-\omega^2 m)-2k^2$$
$$= 3k^2 - 3k\omega^2 m - k\omega^2 M + \omega^4 Mm - 2k^2$$
$$= k^2 - (3km + kM)\omega^2 + Mm\,\omega^4.$$

Quadratic formula:

$$\omega^2 = \frac{k(3m+M) \pm \sqrt{k^2(3m+M)^2 - 4k^2 Mm}}{2k^2}$$

$$= \frac{k(3m+M) \pm k\sqrt{9m^2+M^2+6mM-4Mm}}{2k^2}$$

$$= \boxed{\frac{(3m+M) \pm \sqrt{9m^2+M^2+2Mm}}{2k}}$$

The $\omega=0$ solution no longer exists, so the ~~translational~~ linear motion solution disappears.

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