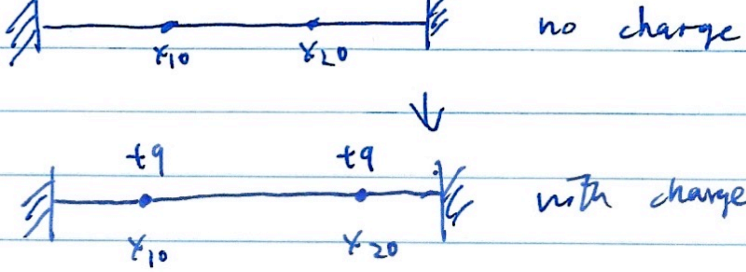
Goldstein
6.13

The force from electrostatics will shift the equilibrium positions slightly :



The potential $V(x_1, x_2)$ is written as

$$V(x_1, x_2) = V_{spring}(x_1, x_2) + V_{coulomb}(x_1, x_2)$$

For small deviations, we expand the potential upto 2^{nd} order:

$$V \approx V\Big|_0 + \frac{\partial V}{\partial x_1}\Big|_0 (x_1 - x_{10}) + \frac{1}{2}\frac{\partial^2 V}{\partial x_1^2}\Big|_0 (x_1 - x_{10})^2$$

$$+ \frac{\partial V}{\partial x_2}\Big|_0 (x_2 - x_{20}) + \frac{1}{2}\frac{\partial^2 V}{\partial x_1 \partial x_2}\Big|_0 (x_1 - x_{10})(x_2 - x_{20})$$

$$+ \frac{1}{2}\frac{\partial^2 V}{\partial x_2^2}\Big|_0 (x_2 - x_{20})^2$$

$$\Rightarrow \quad V \approx \frac{1}{2}\frac{\partial^2 V}{\partial x_1^2}\Big|_0 (x_1 - x_{10})^2 + \frac{1}{2}\frac{\partial^2 V}{\partial x_1 \partial x_2}\Big|_0 (x_1 - x_{10})(x_2 - x_{20}) + \frac{1}{2}\frac{\partial^2 V}{\partial x_2^2}\Big|_0 (x_2 - x_{20})^2$$

$$\equiv \frac{\partial_1^2 V|_0 \, \eta_1^2}{2} + \frac{\partial_2 \partial_1 V|_0 \, \eta_1 \eta_2}{2} + \frac{\partial_2^2 V|_0 \, \eta_2^2}{2}$$

$$V_{spring}(\eta_1,\eta_2) = \frac{1}{2}k\left[a-(x_{10}+\eta_1)\right]^2 + \frac{1}{2}k\left[a-(x_{20}-x_{10})-(\eta_2-\eta_1)\right]^2 + \frac{1}{2}k\left[2a-(x_{20}+\eta_2)\right]^2$$

$$V_{coulomb}(\eta_1,\eta_2) = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(x_{20}+\eta_2)-(x_{10}+\eta_1)}$$

$$\left.\frac{\partial^2 V_{spring}}{\partial \eta_1^2}\right|_0 = 2k, \quad \left.\frac{\partial^2 V_{spring}}{\partial \eta_1 \partial \eta_2}\right|_0 = -k, \quad \frac{\partial^2 V_{spring}}{\partial \eta_2^2} = 2k$$

$$\left.\frac{\partial^2 V_{coulomb}}{\partial \eta_1^2}\right|_0 = \frac{1}{2\pi\varepsilon_0}\frac{q^2}{(x_{20}-x_{10})^3} = \left.\frac{\partial^2 V_{coulomb}}{\partial \eta_2^2}\right|_0 = -\left.\frac{\partial^2 V_{coulomb}}{\partial \eta_1 \partial \eta_2}\right|_0$$

$$\Rightarrow V \approx \left[k + \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(x_{20}-x_{10})^3}\right]\eta_1^2 + \left[k + \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(x_{20}-x_{10})^3}\right]\eta_2^2 + \left[-\frac{k}{2} - \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(x_{20}-x_{10})^3}\right]\eta_1\eta_2$$

As usual, $T = \frac{1}{2}m\dot{\eta}_1^2 + \frac{1}{2}m\dot{\eta}_2^2$,

Let $r_0 \equiv x_{20} - x_{10}$, $\varepsilon \equiv \frac{1}{2\pi\varepsilon_0}$, we have the secalar eq:

$$\begin{bmatrix} 2k + \frac{\varepsilon q^2}{r_0} - \omega^2 m & -\frac{k}{2} - \frac{\varepsilon q^2}{2r_0} \\ -\frac{k}{2} - \frac{\varepsilon q^2}{2r_0} & 2k + \frac{\varepsilon q^2}{r_0} - \omega^2 m \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0.$$

Davidson Cheng

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