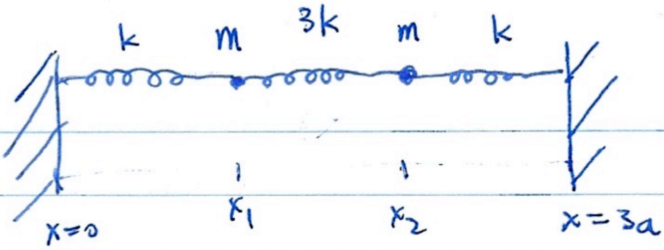
Goldstein 6.12



equilibrium: $x_1 = a$, $x_2 = 2a$.

$$\eta_1 = x_1 - x_{10} = x_1 - a, \quad \eta_2 = x_2 - x_{20} = x_2 - 2a$$

$$V = \frac{k}{2}\left[(x_1-a)^2 + (2a-x_2)^2\right] + \frac{3k}{2}\left[(x_2-x_1-a)^2\right]$$

$$= \frac{k}{2}\left[\eta_1^2 + \eta_2^2\right] + \frac{3k}{2}\left[(\eta_2-\eta_1)^2\right]$$

$$= \frac{k}{2}\left[\eta_1^2+\eta_2^2\right] + \frac{3k}{2}\left[\eta_1^2+\eta_2^2-2\eta_1\eta_2\right]$$

$$T = \frac{m}{2}\left[\dot{x}_1^2+\dot{x}_2^2\right] = \frac{m}{2}\left[\dot{\eta}_1^2+\dot{\eta}_2^2\right]$$

Ansatz: $\vec{\eta} = \begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} = \begin{bmatrix} a_1 e^{-i\omega t} \\ a_2 e^{-i\omega t}\end{bmatrix}$

From Lagrangian $L = \frac{1}{2}(T_i \dot{\eta}_i^2 - V_{ij}\eta_i\eta_j)$, we have the eqm:

$$T_i \ddot{\eta}_i + V_{ij}\eta_j = 0 \qquad \text{(Goldstein 6.10)}$$

$$V = \frac{4k}{2}\eta_1^2 + \frac{4k}{2}\eta_2^2 - \frac{6k}{2}\eta_1\eta_2 = \frac{1}{2}\begin{bmatrix}4k & -3k \\ -3k & 4k\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}$$

$$T = \frac{1}{2}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \dot\eta_1 \\ \dot\eta_2 \end{bmatrix}$$

The eqm $\quad T\ddot{\vec\eta} + V\vec\eta = 0 \quad$ thus gives

$$[V - \omega^2 T]\vec\eta = 0$$

$$\left\{\frac{1}{2}\begin{bmatrix} 4k & -3k \\ -3k & 4k \end{bmatrix} - \omega^2\frac{1}{2}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\right\}\vec\eta = 0.$$

factor out $e^{-i\omega t}$:

$$\left\{\begin{bmatrix} 4k & -3k \\ -3k & 4k \end{bmatrix} - \omega^2\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\right\}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0.$$

$$\begin{bmatrix} 4k-\omega^2 m & -3k \\ -3k & 4k-\omega^2 m \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0.$$

demand $\begin{vmatrix} 4k-\omega^2 m & -3k \\ -3k & 4k-\omega^2 m \end{vmatrix} = 0 \Rightarrow [4k-\omega^2 m]^2 - 9k^2 = 0.$

$$16k^2 + \omega^4 m^2 - 8k\omega^2 m - 9k^2 = 0$$

$$(m^2)(\omega^2)^2 - 8km(\omega^2) + 7k^2 = 0.$$

factoring $\Rightarrow$ $\quad (m\omega^2 - 7k)(m\omega^2 - k) = 0$

$m\omega_1^2 = 7k$, $m\omega_2^2 = k$

$$\boxed{\omega_1 = \sqrt{\frac{7k}{m}} \qquad \omega_2 = \sqrt{\frac{k}{m}}}$$ are the eigenfrequencies.

To find normal modes $\vec{\zeta}$, use $\vec{\eta} = \vec{A}\vec{\zeta}$.

To find $\vec{A}$, solve the eigenvectors:

For $\omega_1 = \sqrt{\frac{7k}{m}}$:

$$\begin{bmatrix} 4k - 7k & -3k \\ -3k & 4k - 7k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

solved via $a_1 = -a_2$, normalization condition yields $a_1 = \frac{1}{\sqrt{2}}$, $a_2 = -\frac{1}{\sqrt{2}}$

For $\omega_2 = \sqrt{\frac{k}{m}}$:

$$\begin{bmatrix} 4k - k & -3k \\ -3k & 4k - k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

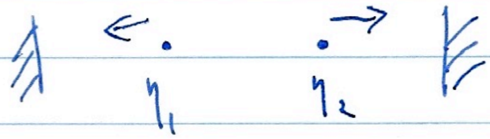
$\Rightarrow$ $a_1 = a_2 = \frac{1}{\sqrt{2}}$.

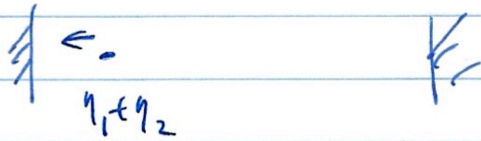
$$\vec{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \vec{A}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\vec{\zeta} = \vec{A}^{-1}\vec{\eta} = \begin{bmatrix} \frac{1}{\sqrt{2}}\eta_1 - \frac{1}{\sqrt{2}}\eta_2 \\ \frac{1}{\sqrt{2}}\eta_1 + \frac{1}{\sqrt{2}}\eta_2 \end{bmatrix}$$

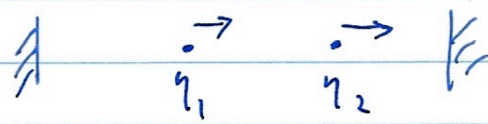
Physical interpretation of the 2 normal modes:

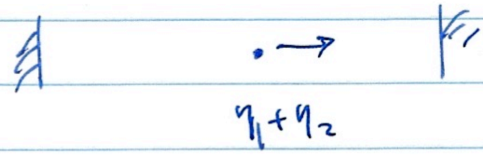
$\frac{1}{\sqrt{2}}\eta_1 - \frac{1}{\sqrt{2}}\eta_2$:





$\frac{1}{\sqrt{2}}\eta_1 + \frac{1}{\sqrt{2}}\eta_2$:





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