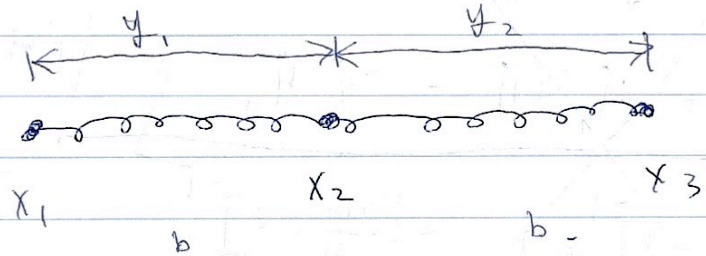
Goldstein

6.1. $V = \frac{k}{2}(y_1 - b)^2 + \frac{k}{2}(y_2 - b)^2$

Let $\eta_1 = y_1 - y_{01}$, $\eta_2 = y_2 - y_{02}$.

Clearly, $y_{01} = y_{02} = b$ in the following configuration



$\eta_1 = y_1 - b$, $\eta_2 = y_2 - b$.

$V = \frac{k}{2}(\dot{\eta}_1^2 + \dot{\eta}_2^2)$.

The "old" form for T is

$$T = \frac{m}{2}[\dot{x}_1^2 + \dot{x}_3^2] + \frac{M}{2}\dot{x}_2^2$$

We need an expression for $\dot{x}_2^2$. Since the COM is at rest,

$$m(\dot{x}_1 + \dot{x}_3) = -M\dot{x}_2,$$

$$\dot{\eta}_1 - \dot{\eta}_2 = 2\dot{x}_2 - (\dot{x}_1 + \dot{x}_3).$$

$$\dot{x}_1 + \dot{x}_3 = 2\dot{x}_2 - (\dot{\eta}_1 - \dot{\eta}_2).$$

$$-\frac{M}{m}\dot{x}_2 - 2\dot{x}_2 = -(\dot{\eta}_1 - \dot{\eta}_2)$$

$$\left(\frac{M}{m}+2\right)\dot{x}_2 = \dot{\eta}_1 - \dot{\eta}_2,$$

$$\dot{x}_2 = \left(\frac{M}{m}+2\right)^{-1}(\dot{\eta}_1 - \dot{\eta}_2).$$

For $\dot{x}_1^2+\dot{x}_3^2$ in T, we use

$$\dot{x}_1^2+\dot{x}_3^2 = \frac{1}{2}\left[(\dot{x}_1+\dot{x}_3)^2 + (\dot{x}_1-\dot{x}_3)^2\right]$$

$$= \frac{1}{2}\left[\frac{M^2}{m^2}\dot{x}_2^2 + (\dot{\eta}_1+\dot{\eta}_2)^2\right].$$

Thus $$T = \frac{m}{4}\left[\frac{M^2}{m^2}\dot{x}_2^2 + (\dot{\eta}_1+\dot{\eta}_2)^2\right] + \frac{M}{2}\dot{x}_2^2.$$

$$= \frac{M^2}{4m}\dot{x}_2^2 + \frac{M}{2}\dot{x}_2^2 + \frac{m}{4}(\dot{\eta}_1+\dot{\eta}_2)^2$$

$$= \frac{M}{4}\left(\frac{M}{m}+2\right)\dot{x}_2^2 + \frac{m}{4}(\dot{\eta}_1+\dot{\eta}_2)^2.$$

$$= \frac{M}{4}\left(\frac{M}{m}+2\right)\left(\frac{M}{m}+2\right)^{-2}\underbrace{\cancel{\dot{x}_2^2}}_{(\dot{\eta}_1-\dot{\eta}_2)^2} + \frac{m}{4}(\dot{\eta}_1+\dot{\eta}_2)^2.$$

$$= \frac{M}{4}\left(\frac{M}{m}+2\right)^{-1}(\dot{\eta}_1-\dot{\eta}_2)^2 + \frac{m}{4}(\dot{\eta}_1+\dot{\eta}_2)^2.$$

$$\Rightarrow T = \frac{1}{2}\left[\frac{M}{2}\left(\frac{M}{m}+2\right)^{-1}(\dot{\eta}_1^2+\dot{\eta}_2^2-2\dot{\eta}_1\dot{\eta}_2) \right.$$
$$\left. + \frac{m}{2}(\dot{\eta}_1+\dot{\eta}_2+2\dot{\eta}_1\dot{\eta}_2\right]$$

Putting it in matrix form:

$$\vec{T} = \begin{bmatrix} \frac{M}{2}\gamma + \frac{m}{2} & \frac{m}{2} - \frac{M}{2}\gamma \\ \frac{m}{2} - \frac{M}{2}\gamma & \frac{M}{2}\gamma + \frac{m}{2} \end{bmatrix},$$

where $\gamma = \left(\frac{M}{m} + 2\right)^{-1}$.

It's obvious that $\vec{V} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Thus $\vec{V} - w^2 \vec{T} = \begin{bmatrix} k - w^2\left(\frac{M}{2}\gamma + \frac{m}{2}\right) & w^2\left(\frac{M}{2}\gamma - \frac{m}{2}\right) \\ w^2\left(\frac{M}{2}\gamma - \frac{m}{2}\right) & k - w^2\left(\frac{M}{2}\gamma + \frac{m}{2}\right) \end{bmatrix}.$

$|\vec{V} - w^2\vec{T}| = 0$ demands.

$$\left[k - w^2\left(\frac{M}{2}\gamma + \frac{m}{2}\right)\right]^2 - \left[w^2\left(\frac{M}{2}\gamma - \frac{m}{2}\right)\right]^2 = 0,$$

$$\left[k - w^2\left(\frac{M}{2}\gamma + \frac{m}{2}\right) - w^2\left(\frac{M}{2}\gamma - \frac{m}{2}\right)\right]\left[k - w^2\left(\frac{M}{2}\gamma + \frac{m}{2}\right) + w^2\left(\frac{M}{2}\gamma - \frac{m}{2}\right)\right] = 0$$

The 2 solutions are.

$$k - w^2\left[\frac{M}{2}\gamma + \frac{m}{2} + \frac{M}{2}\gamma - \frac{m}{2}\right] = 0 \quad \text{and}$$

$$k - w^2\left[\frac{M}{2}\gamma + \frac{m}{2} - \frac{M}{2}\gamma + \frac{m}{2}\right] = 0,$$

They give $w_1 = \sqrt{\frac{k}{M\gamma}} = \sqrt{\frac{k}{M}\left(\frac{M}{m} + 2\right)} = \boxed{\sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}}$

$$\boxed{w_2 = \sqrt{\frac{k}{m}}}$$

These 2 solutions are equivalent to the solutions given in Goldstein 6.4.

The ~~ne~~ zero frequency solution in Goldstein 6.4 corresponds to constant velocity motion of COM. Since we have assumed the COM is at rest at the start of the problem, we have thrown away that solution, and what's left are the two solutions that correspond to relative motion of the particles about the COM.

Davidson Cheng

12.29.2023.