Jackson 6.5 (a)

$\vec{E} = -\vec{\nabla}\Phi, \qquad \vec{\nabla}\times\vec{H} = \vec{J}.$

(6.117) tells us $\vec{P} = \frac{1}{c^2}\int_V \vec{E}\times\vec{H}\, d^3x$

$$= \frac{1}{c^2}\int_V (-\vec{\nabla}\Phi)\times\vec{H}\, d^3x$$

vector quantity: $\vec{\nabla}\times(\psi\vec{a}) = (\vec{\nabla}\psi)\times\vec{a} + \psi(\vec{\nabla}\times\vec{a})$

$\Rightarrow \quad (-\vec{\nabla}\Phi)\times\vec{H} = \Phi(\vec{\nabla}\times\vec{H}) - \vec{\nabla}\times(\Phi\vec{H}),$

$$\vec{P} = \frac{1}{c^2}\int_V \left[\Phi(\vec{\nabla}\times\vec{H}) - \vec{\nabla}\times(\Phi\vec{H})\right] d^3x$$

$$= \boxed{\frac{1}{c^2}\int_V \Phi\vec{J}\, d^3x} - \frac{1}{c^2}\int_V \vec{\nabla}\times(\Phi\vec{H})\, d^3x.$$

The boxed term is prevalent provided $\int_V \vec{\nabla}\times(\Phi\vec{H})\, d^3x$ is small.

$$\vec{\nabla}\times(\Phi\vec{H}) = \partial_1(\Phi H_2) - \partial_2(\Phi H_1)\ \hat{z}$$
$$+ \partial_2(\Phi H_3) - \partial_3(\Phi H_2)\ \hat{x}$$
$$+ \partial_3(\Phi H_1) - \partial_1(\Phi H_3)\ \hat{y}.$$

$$\Rightarrow \left[\vec{\nabla}\times(\Phi\vec{H})\right]_1 = \partial_2(\Phi H_3) - \partial_3(\Phi H_2)$$

$$\int_V \left[\vec{\nabla}\times(\Phi\vec{H})\right]_1 d^3x = \int [\Phi H_3]\Big|_y\, dx\,dz - \int [\Phi H_2]\Big|_z\, dx\,dy.$$

This quantity is small provided $[\Phi H_3]\Big|_y$, $[\Phi H_2]\Big|_z$ are small

The same argument extends to $\int_V [\vec{\nabla} \times (\Phi \vec{H})]_{2,3} \, d^3x$,

so $\Phi \vec{H}$ small at boundary of V makes $\int_V \vec{\nabla} \times (\Phi \vec{H}) d^3x$ small,

which makes

$$\boxed{\vec{P} \simeq \frac{1}{c^2} \int_V \Phi \vec{J} \, d^3x}$$

To make the argument more precise, suppose $[\Phi H_3]|_y$ is

some slowly varying quantity that we can take to be constant,

further we take the volume V to be like a sphere, then

$$\int_V [\Phi H_3]|_y \, dx dy \sim [\Phi H_3]|_y (R^2)$$

If we want this to ~~vanish~~ decrease like $\frac{1}{R}$, we need

$[\Phi H_3]|_y$ to go like $\frac{1}{R^3}$. This means $\Phi \vec{H}(x,y,z)$ should

go like ~~$1/R^3$ where~~ $\frac{1}{x^3}$, $\frac{1}{y^3}$, $\frac{1}{z^3}$,

$$\boxed{\Phi \vec{H}|_x \sim \frac{1}{x^3}, \quad \Phi \vec{H}|_y \sim \frac{1}{y^3}, \quad \Phi \vec{H}|_z \sim \frac{1}{z^3}.}$$

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