Jackson 6.4(a)

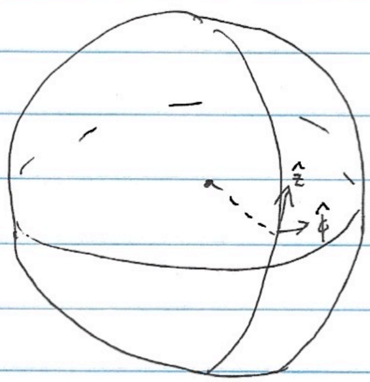
Ohm's law reads: $\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})$

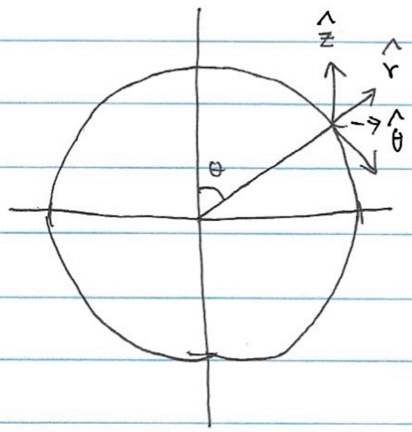
For $\vec{J} = 0$, we have $-\vec{E} = \vec{v} \times \vec{B}$

Inside a uniformly polarized sphere with polarization density $\vec{M} = M\hat{z}$,
The magnetic field inside the sphere is given by $\vec{B} = \frac{2}{3}\mu_0 \vec{M}$

$\vec{v}(r,\theta) = r\sin\theta\,\omega\,\hat{\phi}$

$\vec{v} \times \vec{B} = r\sin\theta\,\omega\left(\frac{2}{3}\right)\mu_0 M(\hat{\phi} \times \hat{z})$





$\hat{\phi} \times \hat{z} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$

$$\Rightarrow \boxed{\vec{E} = \frac{2}{3}\mu_0 M \omega r \sin\theta\,(-1)(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta})}$$

This is the induced electric field.

To find the induced charge density, do $\varepsilon_0 \vec{\nabla} \cdot \vec{E}$:

$$E_r = -\frac{2}{3}\mu_0 M r \omega \sin^2\theta$$

$$E_\theta = -\frac{2}{3}\mu_0 M r \omega \sin\theta\cos\theta$$

$$\vec{\nabla}\cdot\vec{E} = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right)}_{} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, E_\theta\right)}_{}$$

$$-2\mu_0 M r \omega \sin^2\theta$$

$$\left[-\frac{2}{3}\mu_0 M r\omega\right]\left[\frac{1}{r\sin\theta}\right]\left[\frac{\partial}{\partial\theta}\left(\sin^2\theta\cos\theta\right)\right]$$

$$= -\frac{2}{3}\mu_0 M\omega \frac{1}{\sin\theta}\left[2\sin\theta\cos^2\theta - \sin^3\theta\right]$$

$$= -\frac{2}{3}\mu_0 M\omega\left[2\cos^2\theta - \sin^2\theta\right]$$

$$\vec{\nabla}\cdot\vec{E} = -\mu_0 M\omega\left[\frac{4}{3}\cos^2\theta \overset{-}{\cancel{+}} \frac{2}{3}\sin^2\theta + \frac{6}{3}\sin^2\theta\right]$$

$$= -\mu_0 M\omega\left[\frac{4}{3}\right]$$

$$\varepsilon_0\vec{\nabla}\cdot\vec{E} = -\varepsilon_0\mu_0 M\omega\left(\frac{4}{3}\right) = -\frac{4}{3}\frac{M\omega}{c^2} = \rho$$

Substitute $M = \frac{m\,3\,\cancel{R}}{4\pi R^3}$, $\rho = -\frac{\cancel{4}}{\cancel{3}}\frac{\omega}{c^2}\frac{\cancel{3}m}{\cancel{4}\pi R^3}$

$$= \boxed{-\omega m / c^2 \pi R^3}$$

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